A Q-Theory of Inequality[*]

Matthieu Gomez[†]   Émilien Gouin-Bonenfant[‡]

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Abstract

We study the long-run effect of interest rates on wealth inequality. While low rates decrease the average growth rate of existing fortunes, we argue that they *increase* the growth rate of new fortunes by making it cheaper to raise capital. To understand which effect dominates, we derive a sufficient statistic for the effect of interest rates on the Pareto exponent of the wealth distribution: it depends on the average equity issuance rate and leverage of individuals reaching the right tail of the distribution. We estimate this sufficient statistic using new data on the trajectory of top fortunes in the U.S. We conclude that the secular decline in discount rates has played a key role in the recent increase of top wealth inequality.

1   Introduction

Since the seminal contribution of Wold and Whittle (1957), a widespread view is that high interest rates increase top wealth inequality. The intuition is that high rates increase the growth rate of existing fortunes relative to the economy (see Piketty and Zucman, 2015). Yet, this view appears to be at odds with the data: wealth inequality increased substantially in the past forty years in the U.S., a period marked by declining discount rates.

In this paper, we show that lower interest rates can actually *increase* top wealth inequality. Even though lower rates decrease the average growth rate of existing fortunes, they increase

1

the growth rate of new fortunes by making it cheaper for successful entrepreneurs to raise capital.

To be more concrete, consider the trajectory of successful entrepreneurs making it to the top of the wealth distribution. To finance the growth of their firms, these entrepreneurs typically raise capital from outside investors. Lower rates *increase* the rate of capital accumulation of entrepreneurs, since they get less diluted over time. On the other hand—as emphasized by the existing literature—lower rates decrease the rate of capital accumulation of outside investors, who purchase shares of growing firms at elevated prices.

We show that the overall effect of lower rates on the "thickness" of the right tail of wealth distribution depends on the types of individuals making it to the top. If, as in the U.S., individuals at the top of the wealth distribution made their fortunes as entrepreneurs, rather than investors, lower rates increase top wealth inequality.

We use a sufficient statistic approach to quantify the effect of lower rates on top wealth inequality. Our preferred estimate is that a permanent one percentage point decline in discount rates generates a decrease in the Pareto exponent of the wealth distribution by −2.7%. To put this estimate into perspective, this suggests that the 4% decline in discount rates over the 1985-2015 period can account for roughly two-thirds of the decline in the Pareto exponent during the time period. Importantly, this mechanism is consistent with recent studies documenting that the rise in top wealth shares is not driven by the growth rate of existing fortunes, but, rather, by the arrival of new fortunes in top percentiles (e.g., Gomez, 2019; Zheng, 2019).

Overview of the paper. Our paper proceeds in four parts. First, we formalize our idea in a simple stylized model. Entrepreneurs are born with trees. Trees require a continuous flow of investment to grow. To finance the growth of their tree, entrepreneurs continuously sell shares to outside investors ("rentiers"). With some hazard rate, trees blossom and generate a one-time dividend proportional to their size. Afterward, entrepreneurs become rentiers themselves and invest their wealth in a diversified portfolio of trees.

In this stylized economy, we show that Pareto inequality is a U-shaped function of the interest rate.[1] When the interest rate is high, only rentiers make it to the right tail of the distribution. In this case, lower rates decrease Pareto inequality, since it decreases the growth rate of rentiers, as in Wold and Whittle (1957) and Piketty (2015). However, as the interest rate continues to decrease, successful entrepreneurs with growing trees start to reach the right tail of the distribution. When this happens, a further decline in rates *increases* Pareto inequality,

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[1]Pareto inequality is defined as the inverse of the Pareto exponent. When the Pareto exponent is low (the right tail of the wealth distribution is thick), we say that Pareto inequality is high.

since it increases the growth rate of entrepreneurs.

Second, we build a more general model of wealth accumulation to quantify the effect of lower rates on the Pareto exponent of the wealth distribution. Entrepreneurs are now endowed with heterogeneous firms whose productivity evolves according to a Markov chain with an arbitrary transition matrix. Firm growth is determined optimally: capital investment is chosen as to maximize the firm's value subject to adjustment costs. Finally, firms now issue a mix of equity and debt claims to finance their growth.

Despite this arbitrary degree of heterogeneity, we can still derive an analytical expression for the effect of the interest rate on the Pareto exponent of the wealth distribution. Using insights from large deviation theory, we show that it depends on effect of the interest rate on the wealth *trajectory* of individuals making it to the top of the wealth distribution. Put differently, it depends on the effect of the interest rate on the *past* growth rate of individuals at the top of the wealth distribution, not on their *current* growth rate.

In turn, the effect on the interest rate on these trajectories can be expressed in terms of a few observable moments. In particular, it depends on the lifetime average equity payout yield (i.e., the difference between the dividend yield and the equity issuance yield), duration (i.e., the sensitivity of firm value to the interest rate), as well as leverage of the firms owned by individuals making it to the top of the wealth distribution Intuitively, if top individuals relied on a lot of external financing to grow their firm, we expect lower rates to have a large effect on Pareto inequality.

Third, we use new data on the trajectory of top fortunes to estimate these moments for the top 100 individuals in the U.S. We find that the lifetime average equity payout yield of firms owned by top individuals is around −2% annually, which means that entrepreneurs at the top tend to be net equity issuers. The distribution of equity payout yields is extremely skewed: some entrepreneurs own firms with a lifetime average equity payout yield as low as −10%. Moreover, our data reveals an average debt-to-market-equity of about 0.5 (i.e., the market value of the firm exceeds the market value of the equity). Leverage is small for firms backed by venture capital (VC) funding, but it is relatively important for private firms that never raise equity over their lifetime.

Plugging these estimates into our sufficient statistic, we find that the effect of interest rates on wealth inequality is large. According to our preferred measure, a 1 percentage point discount rate decline (i.e., real interest rate plus equity premium) increases Pareto inequality by 2.7%. To put into perspective, we estimate that Pareto inequality has increased by roughly 15% over the 1985 to 2015 period, while discount rates have decreased by roughly 4%. A back-of-

the-envelope calculation suggests that declining discount rates can account for roughly two-thirds of the recent rise in Pareto inequality.

Finally, while our sufficient statistic approach speaks to the long-run effect of lower rates on Pareto inequality, we also examine the dynamics of the wealth distribution along a transition path. To do so, we conduct a simple calibration exercise using the empirical evidence described earlier and simulate the dynamics of the top 1% wealth share in response to an unanticipated, gradual decline in the interest rate (i.e., a sequence of MIT shocks). As the interest rate unexpectedly decreases, the wealth of existing entrepreneurs in the top percentile initially increases due to high realized return. This effect, however, is short-lived, since they earn lower average returns going forward. A second, more important, "displacement" effect sustains the growth of the top 1% wealth share, even in the long-run. Lower rates generate a permanent increase in the flow of entrepreneurs who reach the top percentiles of the wealth distribution, continuously displacing existing fortunes.

Related literature. There is a large body of evidence documenting a rise in top wealth inequality in the U.S. since the 1980s (e.g., Saez and Zucman, 2016; Batty et al., 2019; Smith et al., 2020). A growing literature seeks to understand the factors behind this phenomenon. One strand of the literature focuses on the role of the return on wealth for top individuals (Piketty, 2015; Kuhn et al., 2017; Moll et al., 2019; Hubmer et al., 2020).[2] Another strand of the literature stresses the role of return dispersion (Benhabib et al., 2011; Bach et al., 2015; Fagereng et al., 2020; Benhabib et al., 2019; Atkeson and Irie, 2020). Our findings suggest that these two factors can not be studied in isolation: a decrease in the *average* return on wealth directly affects the *dispersion* of realized returns, which, ultimately, can increase top wealth inequality.[3]

Our mechanism is consistent with a growing body of empirical evidence documenting the fact that the rise in top wealth inequality is driven by the rise of new fortunes, rather than the high growth rates of existing fortunes (Bach et al., 2017; Campbell et al., 2019; Gârleanu and Panageas, 2017; Gomez, 2019; Zheng, 2019).

Our characterization of the Pareto exponent of the wealth distribution builds on the literature on random growth processes (Wold and Whittle, 1957; Acemoglu and Robinson, 2015; Jones, 2015). Recently, this literature has moved towards models with *persistent* growth rate

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[2] Hubmer et al., 2020 argue that the decline in tax progressivity has played a key role in increasing the average after-tax return on wealth. Kaymak and Poschke, 2016 also emphasize the importance of the decline in tax progressivity.

[3] The assumption that idiosyncratic shocks are exogenous to interest rates is key for Piketty and Zucman (2015)'s conclusion. As they observe: "Many different kinds of individual-level random shocks play an important role in practice, and it is difficult to estimate the relative importance of each of them. One robust conclusion, however, is that for *a given variance of shocks*, steady-state wealth concentration is always a rising function of $r - g$." [emphasis added]

heterogeneity (Luttmer, 2011; Jones and Kim, 2016; Benhabib et al., 2015; Gabaix et al., 2016). In this case, the Pareto exponent can be obtained as the principal eigenvalue of an operator related to the transition matrix between states (see de Saporta, 2005 and Beare et al., 2020). Relative to that literature, a theoretical contribution of our paper is to obtain a closed-form expression for the derivative of the Pareto exponent with respect to a parameter (here, the interest rate). We show that the derivative of the Pareto exponent with respect to a parameter depends on its effect on the whole wealth trajectory of individuals reaching the top of the wealth distribution. Remarkably, the effect of interests on the wealth accumulation trajectory of individuals making it to the top in our model depends on a set of moments that can be estimated empirically. This "sufficient statistic" approach allows us to quantify the effect of interest rates on Pareto inequality in a transparent manner.

Beyond the literature on top wealth inequality, several papers examine the redistributive effect of changes in the interest rate. Gârleanu and Panageas (2017), Gârleanu and Panageas (2019) and Kogan et al. (2020) build models in which lower discount rates benefit entrepreneurs at the expense of households. Auclert (2019) stresses the heterogeneous exposure of households to transitory changes in the interest rate.

Our model also relates to the literature on entrepreneurial wealth accumulation (e.g., Quadrini, 2000, Cagetti and De Nardi, 2006; Moll, 2014; Guvenen et al., 2019; Peter, 2019; Tsiaras, 2019). As in these papers, we assume that entrepreneurs remain exposed to their firms, which plays an important role in shaping the wealth distribution. One key difference is that we consider a model where firms can freely issue equity. In our model, as in the data, the most successful firms continuously raise equity and, therefore, vastly outgrow their founder. This allows us to obtain two channels by which lower rates can increase top wealth inequality: the "leverage" effect (for entrepreneurs who issue debt) and the "dilution" effect (for entrepreneurs who issue equity). In the data we find that the dilution effect is most important for VC-backed firms while the leverage effect is most important for private firms who never raise equity.

This focus on equity issuance relates our paper to a large literature studying how firms raise capital. This includes VC funding (Cochrane, 2005; Hall and Woodward, 2010; Opp, 2019; Gornall and Strebulaev, 2015; Gornall and Strebulaev, 2020), equity-based compensation (Ofek and Yermack, 2000; Frydman and Jenter, 2010; Ai et al., 2018; Eisfeldt et al., 2019), IPOs (Ritter and Welch, 2002; Pastor and Veronesi, 2005), as well as seasoned equity offering (Fama and French, 2004; Baker and Wurgler, 2006; Boudoukh et al., 2007).

2 Stylized model

In this section, we describe our mechanism in a stylized model of wealth inequality. Our main result is that, in presence of entrepreneurs, Pareto inequality is a U-shaped function of the interest rate.

2.1 Environment

The economy is populated by infinitely-lived agents. Population grows at rate η. There are two types of agents: "entrepreneurs" and "rentiers". All agents are born entrepreneurs and are endowed with a tree. Trees require outside investment to grow until they blossom. To finance the growth of their tree, entrepreneurs sell shares to rentiers. When an entrepreneur's tree blossoms, it produces apples (the *numéraire*) and the dies. The entrepreneur then becomes a rentier, who invest in a diversified portfolio of trees.

Trees. Each tree starts with a size of one and grows at rate g. To grow, the tree requires a flow of investment i proportional to its size. With constant hazard rate δ, the tree blossoms and returns a one-time positive dividend equal to its size. Formally, the instantaneous cash flow $\mathrm{d}D_t$ of a tree (in unit of apples) is given by

$$
\mathrm{d}D_t = \begin{cases} -ie^{gt}\,\mathrm{d}t & \text{if } t < T \\ e^{gt} & \text{if } t = T \end{cases},
$$

where T denotes the stochastic time at which the tree blossoms. We assume that $g - \delta < \eta$ so that trees do not grow faster than the population. We also assume that $i < \delta$ so that trees return a positive amount of dividend in expectation.

Returns. Because the cash flow of the tree is proportional to its size, the value of a tree is also proportional to its size. Denote q the ratio of the value of a tree to its size. The instantaneous return of holding a tree is given by

$$
\frac{\mathrm{d}R_t}{R_t} = \begin{cases} \left(-\dfrac{i}{q} + g\right)\mathrm{d}t & \text{if } t < T \\ \dfrac{1}{q} - 1 & \text{if } t = T \end{cases}. \tag{1}
$$

While the tree is still growing (i.e., $t < T$), the return is the difference between the growth rate of the tree $g\,\mathrm{d}t$ and the relative amount of new shares $i/q\,\mathrm{d}t$ that must be sold to outside

investors to raise $i\,\mathrm{d}t$. This adjustment reflects the extent to which existing shareholders get diluted during a time period $\mathrm{d}t$ due to equity issuance. When the tree blossoms (i.e., $t = T$), the instantaneous return is $1/q - 1$ since the tree (with price q) is transformed into apples (with price 1).

Denote r to be the interest rate, which we take as given for the moment. The price q is pinned down by the fact that the expected return of holding a tree must equal r:

$$r = -\frac{i}{q} + g + \delta\left(\frac{1}{q} - 1\right). \tag{2}$$

To ensure that the price of the tree is finite, we assume $r > g - \delta$, which implies that $q = (-i + \delta)/(r - g + \delta)$. In particular, the price q is a decreasing function of the interest rate r.

While a low interest rate naturally decreases the average return of holding a tree, notice that it increases the return of holding a tree *conditional on it not blossoming*, which is $-i/q + g$. Intuitively, lower rates (i.e., higher valuations), decrease the rate at which existing shareholders get diluted as the tree grows.[4]

Wealth accumulation. Agents have log utility and discount the future at rate ρ, which implies that they optimally consume a constant fraction ρ of their wealth. Our maintained assumption is that entrepreneurs must have all of their wealth invested in their tree. To finance their investment and consumption, entrepreneurs sell shares to rentiers, who hold a diversified portfolio of trees.

Let W_t be the wealth of an individual. The wealth growth of an entrepreneur during a time period $\mathrm{d}t$ is $\mathrm{d}W_t/W_t = \mathrm{d}R_t/R_t - \rho\,\mathrm{d}t$, which is given by

$$\frac{\mathrm{d}W_t}{W_t} = \begin{cases} \left(-\dfrac{i}{q} + g - \rho\right)\mathrm{d}t & \text{if } t < T \\[2mm] \dfrac{1}{q} - 1 & \text{if } t = T \end{cases}, \tag{3}$$

where T denotes the stochastic time at which the tree blossoms.

When the tree blossoms, the entrepreneur becomes a rentier and invests in a diversified portfolio of trees. The wealth of a rentier evolves as

$$\frac{\mathrm{d}W_t}{W_t} = (r - \rho)\,\mathrm{d}t. \tag{4}$$

Notice that the interest rate has opposite effects on the growth rates of entrepreneurs and

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[4]The dilution rate is the difference between the growth rate of the tree g and the return of a tree that keeps on growing $-i/q + g$, which is i/q. Since q is decreasing in r, the dilution rate is increasing in r.

rentiers. While a lower interest rate decreases the growth rate of rentiers, it increases the growth rate of successful entrepreneurs (i.e., those who own trees that keep on growing). This is shown graphically in Figure 1, which plots the total wealth of an entrepreneur with a tree blossoming at date $T = 15$ in a high interest rate economy as well as a low interest rate economy.
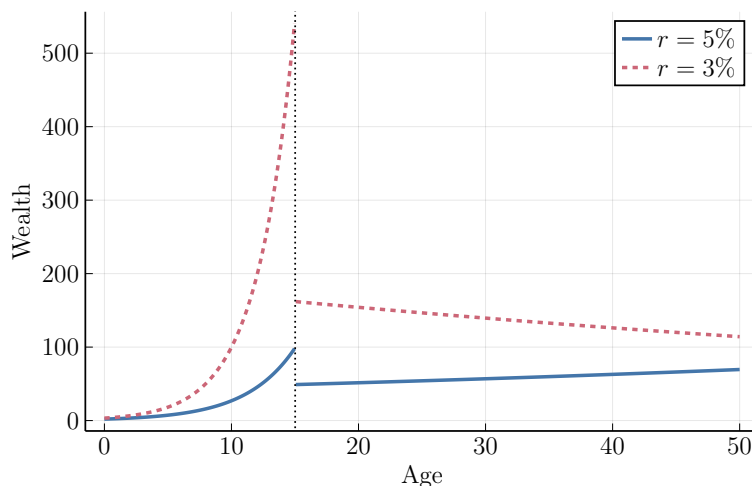

Figure 1: Wealth trajectory of an entrepreneur with a tree that blossoms after 15 years ($T = 15$)

Numerical example with $i = 0.4$, $g = 0.5$, $\delta = 0.5$, $\eta = 0.05$, $\rho = 0.04$

Discussing our assumptions. We now discuss two key assumptions that we made. The first assumption is that trees require outside investment to grow (i.e., $i > 0$). This assumption captures an important characteristic of young firms: they require outside funding to grow. As we will discuss in Section 4, this outside funding is a mix of equity issuance (VC funding or public equity offering), stock-based compensation, and debt financing. The general model—which we present in Section 3—allows firms to have a positive or negative payout yield depending on their current productivity.

The second key assumption is that entrepreneurs must remain fully exposed to their tree. In other words, they must invest all of their wealth in their tree. This assumption captures the fact that most of the wealth of entrepreneurs is invested in their own firm (Quadrini, 2000; Cagetti and De Nardi, 2006; Roussanov, 2010). We take this as exogenous, but this type of portfolio choices can be derived by moral hazard or asymmetric information problems (He and Krishnamurthy, 2012; Brunnermeier and Sannikov, 2014; Di Tella, 2017). In the general model, we will allow entrepreneurs to have an exposure to their firms lower or higher than one (see Footnote 11).

The key distinction between entrepreneurs and rentiers is that entrepreneurs fully invest their wealth in one tree, whereas rentiers own a diversified portfolio of trees. While our model

is very stylized, the term "entrepreneur" should be understood to refer to any individual that is disproportionately exposed to a growing firm. This represents a much larger fraction of the population than strictly defined entrepreneurs. For instance, this includes all the early employees in startups who are paid in stock-options or restricted stocks. Eisfeldt et al. (2019) report that, in recent years, equity-based compensation accounted for 45% of total compensation to high-skilled labor in the U.S. It also includes investors with concentrated portfolios, such as VC investors.

2.2 Wealth distribution

We now characterize the Pareto exponent of the wealth distribution in this economy. We focus on a measure of wealth inequality (i.e., Pareto inequality) which captures the thickness of the right tail of the wealth distribution.

Definition 1 (Pareto inequality). We say that the distribution of a random variable X has a Pareto tail if there exists a $\zeta > 0$ such that

$$\lim_{x \to \infty} \frac{\log \mathbb{P}(X > x)}{\log x} = -\zeta.$$

The parameter ζ is called the Pareto exponent.

A low ζ corresponds to a thick tail (i.e., a density that decays slowly as $w \to \infty$). Following Jones (2015), we define Pareto inequality θ as the inverse of the Pareto exponent (i.e., $\theta = 1/\zeta$). A high level of Pareto inequality thus corresponds to a thick upper tail. The following proposition characterizes the steady-state level of Pareto inequality as a function of the interest rate r.[5]

Proposition 2. *Assume that $\rho < g - i$. Then*

$$\theta = \max \left(\frac{-\frac{i}{q} + g - \rho}{\eta + \delta}, \frac{r - \rho}{\eta} \right).$$

The proposition says that Pareto inequality is pinned down by the maximum of two expressions. The first expression corresponds to the growth rate of successful entrepreneurs divided by the sum of population growth η and the transition rate δ. The second expression corresponds to the growth rate of rentiers divided by population growth.

The first expression is decreasing in r, while the second expression is increasing in r. One can show that there exists $r^* \in (g - \delta, \rho + \eta)$ such that they intersect. Overall, this says that

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[5]To be precise, we consider a balanced growth path where growth per capita is zero.

Pareto inequality is a U-shaped function of the interest rate and can be expressed as

$$\theta = \begin{cases} \dfrac{-\frac{i}{q} + g - \rho}{\eta + \delta} & \text{for } r \in (g - \delta, r^*), \\[2ex] \dfrac{r - \rho}{\eta} & \text{for } r \in (r^*, \rho + \eta). \end{cases} \tag{5}$$

Figure 2 plots the relationship between the interest rate and Pareto inequality in a numerical example.

When $r > r^*$, the right tail of the wealth distribution is only populated by rentiers (i.e., the ratio of the mass of rentiers to the total mass of agents converges to one as wealth goes to infinity).[6] In this case, lower rates decrease Pareto inequality, since the growth rate of rentiers is increasing in the interest rate. Notice that the formula for the Pareto exponent is the same as in an economy with only one type of agent (see Wold and Whittle, 1957 and Piketty and Zucman, 2015).

In contrast, when $r < r^*$, entrepreneurs are present in the right tail of the wealth distribution. As long as this is the case, lower rates *increase* Pareto inequality. This is because the right tail of the distribution is determined by the growth rate of successful entrepreneurs and those agents benefit from lower interest rates. As explained earlier, a decline in the interest rate leads to an increase of the price of a share q. High valuations imply that entrepreneurs can finance the growth of their tree by issuing fewer shares, thereby leading to less dilution.

Notice that, as long as $r < r^*$, Pareto inequality is fully pinned down by the growth rate of entrepreneurs. In that case, the growth rate of rentiers only matters in determining the relative mass of entrepreneurs and rentiers in the right tail (see Corollary 6 in Appendix B). We consider the case $r < r^*$ to be the empirically relevant one. As we will see in Section 4, entrepreneurs account for most of individuals at the top of the wealth distribution in the U.S.

General equilibrium. So far, we have considered an exogenous change in the interest rate. Throughout the paper, we remain agnostic regard the exact source of the change in the interest rate. For instance, it could come from a change in savings coming from abroad (global savings glut) or from domestic markets (population aging). In Appendix B.3, we solve a general equilibrium extension of our model which incorporates an additional group of agents (i.e., "workers"). We show that varying the subjective discount factor of workers from zero to infinity can generate the range of interest rates considered in Proposition 2.

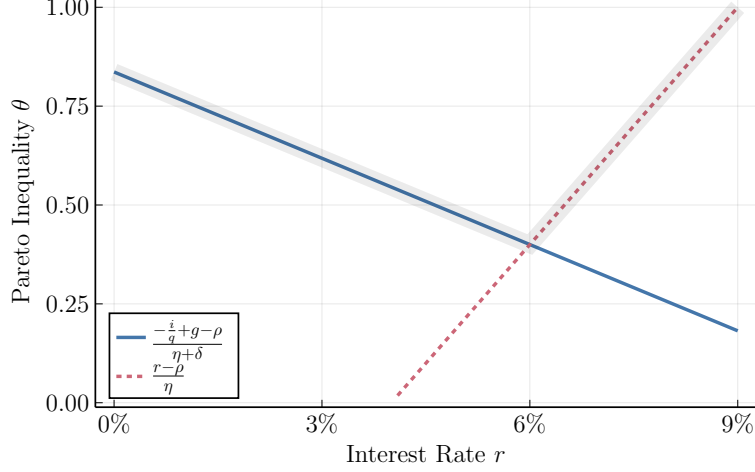---

[6]See Corollary 6 in Appendix B.

Figure 2: Pareto inequality θ as a function of the interest rate.

Numerical example with $i = 0.4$, $g = 0.5$, $\delta = 0.5$, $\eta = 0.05$, $\rho = 0.04$

Sufficient statistic. In our stylized model, we can derive a simple formula for the effect of the interest rate on Pareto inequality. As long we are in the region in which there are entrepreneurs in the right tail of the wealth distribution (i.e., $r < r^*$), we obtain the following "sufficient statistic" for the effect of the interest rate on Pareto inequality:

$$
\begin{aligned}
\partial_r \log \theta &= \partial_r \log \left(-\frac{i}{q} + g - \rho \right), \\
&= \frac{-\frac{i}{q} |\partial_r \log q|}{-\frac{i}{q} + g - \rho}.
\end{aligned}
\tag{6}
$$

This expression depends on three distinct quantities: the payout yield of the tree $-i/q$, the duration of the tree (defined as the sensitivity of its value to the interest rate) $|\partial_r \log q|$, and the growth rate of wealth for successful entrepreneurs $-\frac{i}{q} + g - \rho$. This insight holds in a much more general model, as we show in the next section.

3 General model

The stylized model in the previous section generates clear predictions regarding the effect of lower rates on Pareto inequality. However, it relies on strong simplifying assumptions: (i) all trees grow at the same rate (i.e., no heterogeneity within entrepreneurs); (ii) the growth rate of trees does not depend on the interest rate (i.e., no optimal investment decision); (iii) entrepreneurs only issue equity to finance growth (i.e., no debt issuance). In this section, we build a more general model that relaxes these assumptions.

In this more general model, we derive a sufficient statistic for the derivative of Pareto inequality with respect to the interest rate and show that it can be expressed in terms of ob-

11

servable moments. Overall, we find that the long-run effect of a change in rates on Pareto inequality depends on the *past* equity payout yield, duration, and leverage of entrepreneurs currently the right tail of the wealth distribution.

3.1 Set up

Environment. The economy is populated by a continuum of agents. Population grows at rate η. Agents are born entrepreneurs and are endowed with a firm. The firm has initial capital equal to one.[7] All risk is idiosyncratic. With constant hazard rate δ, entrepreneurs sell their firms and become rentiers who live forever and hold a diversified portfolio with return r.

Firm problem. Firms produce a homogeneous consumption good (the *numéraire*) and operate an aK technology where a denotes TFP and K denotes the capital stock. TFP evolves over time according to a time-reversible Markov Chain with states $s \in \{1, \dots, S\}$.

The problem of a firm in state s is to choose a growth rate g as to maximize the present value of future payouts discounted at rate r. In order to grow its capital stock by gK, the firm must invest $i(g)K$ units of the consumption goods. We assume that the adjustment cost function satisfies $i'(0) = 1$, $i'(\cdot) > 0$ and $i''(\cdot) > 0$. The value of a firm $V_s(K)$ is the solution to the following *Hamilton-Jacobi* equation (HJB):

$$rV_s(K) = \max_g \left\{ (a_s - i(g))K + V_s'(K)gK + (\mathcal{T}V)_s(K) \right\}, \tag{7}$$

where $\mathcal{T} = (\tau_{ss'})$ denotes the transition probability matrix for the Markov state s.[8] Given that the value function is homogeneous in capital K, it can be written as $V_s(K) = q_s K$, where q_s is ratio of the market value of the firm to its book value (i.e., Tobin's q). The HJB thus simplifies to

$$rq_s = \max_g \left\{ a_s - i(g) + q_s g + (\mathcal{T}q)_s \right\}. \tag{8}$$

From now on, we assume that there exists a positive and finite solution $\{q_1, \dots, q_S\}$ to Equation 8. The HJB implies that the optimal growth rate g_s satisfies the following first-order condition:

$$i'(g_s) = q_s. \tag{9}$$

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[7]We could also allow initial capital to be heterogeneously distributed across entrepreneurs. As long as the right tail of this initial distribution is thinner than the tail of the steady-state wealth distribution, this has no impact on our results.

[8]To be precise, $\mathcal{T}$ is the infinitesimal generator for s defined by the action $(\mathcal{T}q)_s = \mathbb{E}_s[\frac{dq_s}{dt}]$. We abuse notation and treat $\mathcal{T}$ as an $S \times S$ matrix.

Return. The return of owning a firm is

$$\frac{dR_t}{R_t} = \frac{a_{s_t} - i(g_{s_t})}{q_{s_t}} dt + g_{s_t} dt + \frac{dq_{s_t}}{q_{s_t}}, \tag{10}$$

where s_t denotes the Markov state of the firm at time t. The return is the sum of three terms: the payout yield $(a_s - i(g_s))/q_s$, the growth rate of the firm g_s, and the change in valuation dq_s/q_s.

For the moment, we assume that firms raise capital through equity issuance and distribute payouts through dividend issuance. When the firm payout is negative (i.e., $i(g_s) > a_s$), the firm pays no dividend and must issue equity in order to finance its investment. In this case, the return of owning the firm is the growth rate of capital g_s minus the dilution rate $(i(g_s) - a_s)/q_s$ plus the change in valuation $\frac{dq_{s_t}}{q_{s_t}}$. When the firm payout is positive (i.e., $i(g_s) < a_s$), the firm distributes the output net of investment expenditures to the owners of the firm in the form of a dividend. The return is thus the growth rate of capital g_s plus the dividend yield $(a_s - i(g_s))/q_s$ plus the change in valuation $\frac{dq_{s_t}}{q_{s_t}}$. The HJB ensures that the expected return is equal to the interest rate (i.e., $\mathbb{E}_t \frac{dR_t}{R_t} = r\,dt$), which pins down the market value of the firm.

3.2 Wealth accumulation

As in the stylized model, agents have log utility and discount the future with rate ρ, which implies that they optimally consume a constant fraction ρ of their wealth. Agents are born with a firm with initial productivity state s drawn from a distribution ψ_0. Entrepreneurs must invest all of their wealth in their firm, while rentiers own a diversified portfolio of firms.

The law of motion for the wealth of an entrepreneur is

$$\frac{dW_t}{W_t} = \frac{dR_t}{R_t} - \rho\,dt.$$

Plugging the expression for the return of the firm (Equation 10) in the law of motion, we obtain

$$\frac{dW_t}{W_t} = \underbrace{\left(\frac{a_{s_t} - i(g_{s_t})}{q_{s_t}} + g_{s_t} - \rho \right)}_{\mu_{s_t}} dt + \frac{dq_{s_t}}{q_{s_t}}. \tag{11}$$

The drift term μ_s corresponds to the rate of capital accumulation of the entrepreneur, while the jump term dq_{s_t}/q_{s_t} accounts for the change in the value of the capital. Note that the steady-state level of Pareto inequality is only determined by the drift term μ_s. We formalize this result in Proposition 3, but the idea is that Pareto inequality is only determined by the distribution

13

of capital across entrepreneurs, not the distribution of its valuation.[9]

What is the effect of a change in the interest rate on the rate of capital accumulation? Differentiating the expression for μ_s in (11), we obtain:

$$\partial_r \mu_s = \underbrace{\frac{a_s - i(g_s)}{q_s}}_{\text{payout yield}} \underbrace{|\partial_r \log q_s|}_{\text{duration}} + \underbrace{\left(1 - \frac{i'(g_s)}{q_s}\right) \partial_r g_s}_{=0}. \tag{12}$$

The derivative of the rate of capital accumulation with respect to the interest rate is the sum of two terms. The first term accounts for the effect of r on the value of the firm, while the second term accounts for the effect of r on the optimal growth rate of capital. Because the optimal growth rate ensures that the firm invests up to the point where the marginal cost of capital equals its marginal value, the second term is zero (i.e., an application of the envelope theorem).[10]

Therefore, the effect of a change in the interest rate on μ_s depends on the payout yield and duration of the firm owned by the entrepreneur. In particular, lower rates increase the growth rate of μ_s in states where the firm is a net equity issuer (i.e., $\frac{a - i(g)}{q} < 0$). Intuitively, when interest rates decline, valuations increase, which means that fewer shares must be sold to outside investors in order to raise capital. This increases the rate at which successful entrepreneurs (i.e., those who own fast-growing firms that rely on external financing) accumulate capital. In contrast, firms respond to a decline in r by investing more, but, at the first order, this additional capital does not benefit the entrepreneur, since its marginal cost is equal to its marginal benefit.

With constant hazard rate δ, the entrepreneur sells her firm and uses the proceeds to invest in a diversified portfolio of firms. The growth rate of wealth for rentiers is simply given by

$$\frac{dW_t}{W_t} = (r - \rho)\, dt. \tag{13}$$

3.3 Leverage

So far, we have considered the case of all-equity firms. In reality, firms issue a mix of equity and debt claims. As we now show, both types of financing matter for the effect of interest rates on the growth rate of entrepreneurs.

We now assume that firms have a constant ratio of debt to equity κ. We assume that entrepreneurs invest all of their wealth in the equity of their firms.[11] Let $q_{\kappa s}$ be the price of an

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[9]Formally, this comes from the fact that the ratio between wealth and the capital owned by the entrepreneur (i.e., Tobin's q) is bounded in our model.

[10]Ozdagli (2018) and Darmouni et al. (2020) use a similar insight to interpret a firm stock market response to monetary policy.

[11]We can also consider the case in which entrepreneurs own levered position in the equity of their firms. Formu-

equity share for a firm in state s. We have

$$q_{\kappa s} = q_s + \kappa(q_s - 1). \tag{14}$$

The law of motion of wealth for an entrepreneur is now given by

$$\frac{dW_t}{W_t} = \underbrace{\left(\frac{(1+\kappa)(a_{s_t} - i(g_{s_t})) - \kappa(r - g_{s_t})}{q_{\kappa s_t}} + g_{s_t} - \rho \right)}_{\mu_{s_t}} dt + \frac{dq_{\kappa s_t}}{q_{\kappa s_t}}. \tag{15}$$

As in the case without leverage, it is the sum of two terms. The first term μ_{s_t} corresponds to the rate at which the entrepreneur accumulates equity capital. It is the sum of the equity payout yield, the growth rate of capital g_s, minus the consumption rate ρ. Notice that the payout to equity holders is now the difference between output net of investment $(1+\kappa)(a_s - i(g_s))$ and the net payout to debt-holders $\kappa(r - g_s)$, both per unit of equity capital. The second term, $dq_{\kappa s_t}/q_{\kappa s_t}$ corresponds to changes in the value of the equity capital.

Equation 15 indicates two channels by which lower rates change the rate of equity capital accumulation for entrepreneurs. First, as in the case without leverage, a lower r increases the value of the firm equity, which reduce the dilution rate of existing shareholders when firms raise outside capital. Second, due to leverage, a lower r increases the amount of cash returned to equity-holders at the expense of debt-holders. Formally, the effect of a change in r on the rate of equity capital accumulation is the sum of two terms:

$$\partial_r \mu_s = \underbrace{\frac{(1+\kappa)(a_s - i(g_s)) - \kappa(r - g_s)}{q_{\kappa s}}}_{\text{equity payout yield}} \underbrace{|\partial_r \log q_{\kappa s}|}_{\text{duration}} - \underbrace{\frac{\kappa}{q_{\kappa s}}}_{\text{debt-to-market equity}}, \tag{16}$$

where duration refers to the semi-elasticity of the value of equity $q_{\kappa s}$ with respect to the interest rate. Both equity and debt issuance matter for the effect of lower rates on the growth rate of entrepreneurs. For example, entrepreneurs who neither raise outside equity nor distribute payouts to equity holders, but use debt to grow, benefit from lower rates. This comes from the fact that lower rates decrease the debt payment to outside investors. Naturally, when firms are not levered (i.e. $\kappa = 0$), Equation 16 coincides with the formula in the case without leverage (i.e., Equation 12).

This distinction between equity and debt requires a discussion of what we call interest rate. In our model, productivity shocks are purely idiosyncratic. Therefore, both equity and debt are discounted at the same rate r. In the real world, however, debt and equity are discounted using

las remain the same, after redefining κ to be the effective debt to equity of entrepreneurs.

different rates. The first term in Equation 16 (equity payout yield times duration) corresponds to the effect of a change in the required return on equity. The second term (debt-to-market equity) corresponds to the effect of a change in the required return on debt. For the rest of the paper, we consider a change in both rates of return, even though one could adapt our framework to only consider a change in, say, the equity risk premium.

3.4 Wealth distribution

The following proposition characterizes the right tail of the steady-state wealth distribution.

Proposition 3 (Pareto tail). *Suppose that there there is at least one productivity state s' such that the rate of capital accumulation $\mu_{s'} > 0$. Then, the distribution of wealth has a Pareto tail with Pareto inequality given by*

$$\theta = \max \left(\theta_E, \frac{r - \rho}{\eta} \right),$$ (17)

where θ_E denotes the unique positive number such that

$$\rho_D \left(\frac{1}{\theta_E} \mathcal{D}(\boldsymbol{\mu}) + \mathcal{T} \right) = \delta + \eta.$$ (18)

We define $\rho_D(\cdot)$ to be the dominant eigenvalue of a matrix and $\mathcal{D}(\boldsymbol{\mu})$ to be the diagonal matrix with elements $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_S)$ on its main diagonal.

As in the stylized model, Pareto inequality is the maximum between θ_E and $\frac{r - \rho}{\eta}$. Moreover, as in the stylized model, θ_E depends on the rate of capital accumulation of entrepreneurs (through the implicit Equation 18). If $\frac{r - \rho}{\eta} > \theta_E$, the distribution of wealth across rentiers has a thicker tail than the distribution of wealth across entrepreneurs, which implies that the tail of the wealth distribution is only populated by rentiers. In contrast, if $\theta_E > \frac{r - \rho}{\eta}$, both types are in the right tail of the wealth distribution (i.e. rentiers "inherit" the Pareto tail of entrepreneurs). As discussed earlier, the first case is the empirically relevant one. From now on, we assume that $\theta_E > \frac{r - \rho}{\eta}$.

Notice that Proposition 3 can be seen as a generalization of Proposition 2 for the stylized model. Indeed, in the stylized model, we have that $\mathcal{T} = 0$ (there is only one productivity state) and $\mathcal{D}(\boldsymbol{\mu}) = -\frac{i}{q} + g - \rho$. Plugging these expressions in (18), we recover the result from Proposition 2.

3.5 Sufficient statistic

Compared to the stylized model, there is no closed-form solution for Pareto inequality θ. Nevertheless, we now show that we can obtain a simple closed-form expression for the derivative of Pareto inequality with respect to the interest rate (i.e., $\partial_r \theta$). This is our main theoretical result.

Proposition 4. *Denote $\boldsymbol{u}, \boldsymbol{v}$ to be the left and right eigenvectors associated with the dominant eigenvalue of the matrix $\frac{1}{\theta}\mathcal{D}(\boldsymbol{\mu}) + \mathcal{T}$, normalized so that $\boldsymbol{u}'\boldsymbol{1} = \boldsymbol{u}'\boldsymbol{v} = 1$. The derivative of Pareto inequality θ with respect to the interest rate is given by:*

$$\partial_r \log \theta = \frac{(\boldsymbol{u} \cdot \boldsymbol{v})' \partial_r \boldsymbol{\mu}}{(\boldsymbol{u} \cdot \boldsymbol{v})' \boldsymbol{\mu}}, \tag{19}$$

where $\cdot$ denotes the element-wise multiplication of two vectors.

In other words, the derivative of the Pareto exponent with respect to r is proportional to the derivative of the rate of capital accumulation, averaged across productivity states using the vector $\boldsymbol{u} \cdot \boldsymbol{v}$. Because $\boldsymbol{u}$ and $\boldsymbol{v}$ correspond to the eigenvectors associated with the dominant eigenvalue, they are positive element-wise. Therefore, $\boldsymbol{u} \cdot \boldsymbol{v}$ is a density on the productivity states. The next proposition shows that this density has a physical interpretation: it corresponds to the density of *past* states for individuals in the right tail of the wealth distribution.[12]

Proposition 5 (Sufficient statistic). *Let τ denote the current age of an individual. The derivative of Pareto inequality θ with respect to the interest rate is given by*

$$\partial_r \log \theta = \lim_{w \to +\infty} \mathrm{E}\left[\left. \frac{\frac{1}{\tau}\int_0^\tau \partial_r \mu_{s_t}\,\mathrm{d}t}{\frac{1}{\tau}\int_0^\tau \mu_{s_t}\,\mathrm{d}t} \right| W = w \right]. \tag{20}$$

This equation says that the effect of the interest rate on the Pareto exponent is given by its effect on the *past* rate of capital accumulation of entrepreneurs currently in the right tail. This form of ex-post conditioning is key: what matters is the effect of interest rates on the growth rate of the entrepreneurs that *are going to reach the top*, not its effect on the growth rate of entrepreneurs *already at the top* (which is always negative).

Plugging the expression for the effect of the interest rate on the rate of equity capital accumulation $\partial_r \mu_s$ (Equation 16) in Equation 20, we obtain a sufficient statistic for the effect of

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[12]We refer the reader to Lecomte (2007) for results on the physical interpretation of left and right eigenvectors of tilted generators in non-stationary environment.

17

interest rates on Pareto inequality in terms of observable moments:

$$\partial_r \log \theta = \lim_{w \to +\infty} \mathbb{E} \left[\left. \frac{\frac{1}{\tau} \int_0^\tau \left(\frac{(1+\kappa)(a_s - i(g_s)) - \kappa(r - g_s)}{q_{\kappa s}} \mid \partial_r \log q_{\kappa s} \mid - \frac{\kappa}{q_{\kappa s}} \right) \mathrm{d}t}{\frac{1}{\tau} \int_0^\tau \mu_{s_t} \mathrm{d}t} \right| W = w \right]. \tag{21}$$

In words, we have

$$\partial_r \log \theta = \lim_{w \to +\infty} \mathbb{E} \left[\left. \frac{\text{equity payout yield} \times \text{duration} - \text{debt-to-market equity}}{\text{growth rate}} \right| W = w \right], \tag{22}$$

where each quantity refers to the average taken over an individual's lifetime.

Extensions. In Appendix D, we extend this sufficient statistic along two dimensions. First, we study the role of constraints on external financing, which we model as a lower bound on the equity payout yield (the case of no-equity issuance is a special case). In this case, the sufficient statistic is similar, except that Equation 21 must be multiplied by one plus the shadow value of relaxing the financing constraint. Intuitively, financing constraints amplify the effect of interest rate on inequality, since lower rates relax the financing constraint.

Second, we consider the case in which households have an arbitrary elasticity of intertemporal substitution (EIS) instead of log utility (which corresponds to EIS $= 1$). In this case, lower rates not only affect the realized returns of entrepreneurs, they also affect their saving rates. If EIS < 1, lower rates increase the saving rate, while when $EIS > 1$, lower rates increase the saving rate. In either case, the sufficient statistic in Equation 21 must be modified by the difference between the EIS and one.

4   Empirics

Given a sample of N individuals in the right tail of the wealth distribution, Equation 22 shows that our sufficient statistic can be estimated as

$$\widehat{\partial_r \log \theta} = \frac{1}{N} \sum_{i=1}^N \frac{\left(\text{equity payout yield} \times \text{duration} \right)_i - \text{debt-to-market equity}_i}{\text{growth rate}_i}, \tag{23}$$

where each quantity refers to the average taken over an individual's lifetime.

While the existing literature focuses on the characteristics of individuals at the top of the wealth distribution (for instance Cagetti and De Nardi, 2006), relatively little is known regarding the *trajectory* of individuals reaching the top. Our main empirical contribution is to con-

struct a database on the growth rate of wealth, equity payout yield, and leverage of individuals reaching the top of the wealth of the wealth distribution.

4.1 Estimating the sufficient statistic

Forbes list. We identify individuals in the right tail of the wealth distribution using the list of the wealthiest 400 Americans produced by Forbes Magazine. The list is created by the staff of the magazine based on a mix of public and private information.[13] For our application, we focus on the year 2015 and define the "right tail" as individuals in the top 100, a group for which information is widely available.

Table 1: Individuals in the top 100 (Forbes list, 2015)

Group	Count
Entrepreneurs	72
Public corporation	42
Private corporation	30
Rentiers	4
Financiers	24

Notes. "Entrepreneurs" are defined as individuals who are invested in non-financial firms that they (or a family member) founded; "Rentiers" are defined as individuals who are no longer invested in the firm that they (or a family member) founded; "Financiers" are defined as individuals who are invested in a financial firm that they (or a family member) founded.

Table 1 contains information on the top 100 individuals included in the Forbes list in 2015. We assign to each individual the main firm that they or their family founded. Out of this set of individuals, we remove 4 "rentiers", which we define as individuals who are no longer invested in the firm that they or their family founded.[14] We also remove 24 "financiers", which we define as individuals who founded a financial firm, since our framework does not directly apply to them. We are left with 72 individuals for which we have detailed information (age, wealth, source of wealth, firms that they founded, etc.). Roughly 60% own public firms while the rest own private firms. Table 5 in Appendix A contains a detailed list of the individuals in our sample.

Equity payout yield. We now construct a measure of the average payout yield of firms owned by individuals in our list. This is not a trivial task because most of the equity issuance happens before the firm becomes public, and, therefore, is not observable in Compustat/CRSP.

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[13]Forbes Magazine reports: "We pored over hundreds of Securities Exchange Commission documents, court records, probate records, federal financial disclosures and Web and print stories. We took into account all assets: stakes in public and private companies, real estate, art, yachts, planes, ranches, vineyards, jewelry, car collections and more. We also factored in debt. Of course, we don't pretend to know what is listed on each billionaire's private balance sheet, although some candidates do provide paperwork to that effect."

[14]As discussed in the context of the stylized model, the Pareto exponent is purely determined by the growth rate of entrepreneurs.

The equity payout yield is the difference between the dividend payout yield (i.e., dividends distributed divided by the market value of the firm equity) and the equity issuance yield (i.e., equity issued minus equity repurchased divided by the market value of the firm equity).

We first estimate the average equity issuance yield, which corresponds to the average growth rate of the number of shares in the firm. For firms that are public in 2015, the average equity yield can be computed as the log ratio between the number of shares in 2015 and the initial number of shares, divided by the age the firm. Formally, we use

$$\text{equity issuance yield} = \frac{\log\left(N_{2015}/N_{t_0}\right)}{2015 - t_0 + 1}, \tag{24}$$

where N_{2015} denotes the number of shares in 2015 (adjusted for stock splits since the IPO) and N_{t_0} denotes the number of shares when the firm was founded. We measure N_{2015} by summing the number of common and reserved shares reported in the 2015 10-K filing. We measure N_{t_0} as the number of shares owned by founders held at the time of the IPO.[15] Both measures are available using the S-1 filing, which is filed at the time of the IPO.

For firms that are not public in 2015, we compute the equity issuance yield as the log of one over the ownership share of founders in 2015 (consolidated at the family level), divided by the age of the firm. For founders of private firms who fully own their firms in 2015, the equity issuance yield is therefore zero. For founders who own only a fraction of their firm in 2015, the equity issuance yield is positive.

We then estimate the dividend yield. For firms who were public in 2015, we compute their annual dividend yield for every post-IPO year as the difference between the log return and the log return excluding dividends from CRSP.[16] We make the assumption that the dividend yield pre-IPO was zero. Overall, we obtain an average annual dividend yield of 1.2%. We use this figure to impute the dividend yield of firms that were private in 2015.

We report summary statistics for the equity payout yield in Table 2. Overall, we find that firms owned by individuals in the right tail in 2015 have had an average annual equity payout yield of -2% since they were founded. The distribution of the equity payout yields is highly negatively skewed, with values ranging from -18% for Travis Kalanick (Uber) to 3% for Les Wexner (L brands).

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[15]In some cases, we adjust this number by the number of shares granted to them in between the founding date and the IPO data as a compensation for their labor, (e.g., Mark Zuckerberg) or shares that they bought during funding rounds (e.g., Elon Musk).

[16]Formally, we compute the dividend yield as $\log(1 + \texttt{ret}) - \log(1 + \texttt{retx})$.

Table 2: Summary Statistics

	Obs.	Average	Percentiles				
			Min	p25	p50	p75	Max
Equity payout yield	72	−0.02	−0.18	−0.02	0.01	0.01	0.03
Debt-to-Market-Equity	72	0.48	0.00	0.22	0.48	0.48	3.68
Growth rate	72	0.32	0.02	0.14	0.21	0.40	1.56

Debt-to-market equity. We estimate the debt-to-market equity as follows. For firms who were public in 2015, we compute the average debt-to-market equity for every post-IPO years using data from Compustat.[17] We find an average debt-to-market equity of 0.48. We use this figure to impute the debt-to-market equity of firms that were private in 2015.

Growth rate of wealth. We estimate the average growth rate of wealth as the log ratio between wealth in 2015 and an imputed initial wealth, divided by the age of the founded firm. Formally, we use

$$\text{growth rate} = \frac{\log\left(W_{2015}/W_{t0}\right)}{2015 - t_0 + 1}, \tag{25}$$

where t_0 denotes the founding date of the firm, W_{2015} denotes the wealth in year 2015, and $W_{t_0} = 100,000$. Our benchmark assumption is that, at the founding date, the founder had $100,000 (in 2015 dollars) in equity capital. We then subtract the average real wealth growth per capita between t_0 and 2015 to obtain the *relative* wealth growth (in our model, wealth growth per capita is zero).[18]

We report summary statistics in Table 2. We estimate an average growth rate of 32%. The distribution of growth rates is positively skewed, with large outliers corresponding to Facebook and Whatsapp founders. Unsurprisingly, heirs of entrepreneurs who founded firms in the distant past tend to have much lower average growth rates.

Duration. The effect of the interest rate on Pareto inequality depends on the average equity duration of the firms owned by individuals that reach the top of the wealth distribution. Equity duration is defined as the derivative of the log price of the firm equity with respect to the interest rate. The higher the duration of a firm, the more its share price responds to a (permanent and unanticipated) changes in interest rates.

Ideally, we would measure firm duration as the reaction of its market value to unexpected

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[17]Formally, we use the formula (at- seq)/me.

[18]We use aggregate net worth (TNWBSHNO) deflated using the CPI (CPIAUCSL) divided by civilian non-institutional population (CPI160V), where all variable are obtained from FRED.

and permanent changes in interest rates. However, this is hard to do empirically. Empirically, unexpected monetary policy shocks correspond to transitory changes in interest rates, and their effect on firm values may be confounded by cash-flow or information effects (Nakamura and Steinsson, 2018).

Given these difficulties, we do not attempt to measure duration for each firm separately and instead impose a constant duration across firms. We adapt estimates from Gormsen and Lazarus (2019), who measure the ex-post duration of firms in CRSP. They find that, while the average firm has a duration of roughly 20 years, the average duration of the top 20% of the firms in CRSP (sorted according to an ex-ante duration measure) is 46 years.[19] In our preferred calibration, we assume a duration of 30 years. We see this as a conservative assumption: most equity issuance occurs before a firm actually becomes mature enough to enter the CRSP sample. For robustness, we also explore two alternative hypothesis: a duration of 20 years and a duration of 40 years.

Results. We now use Equation 23 to combine our estimates of the average equity payout yield, debt-to-market equity, and growth rate of wealth of individuals reaching the top of the wealth distribution. Table 3 contains the predicted long-run effect of the interest rate on Pareto inequality $\widehat{\partial_r \log \theta}$ in our preferred calibration as well as using an alternative hypothesis about duration. To describe the variation in the data, we also compute the sufficient statistic for each individual and report summary statistics. In our preferred calibration, we obtain a value for $\partial_r \log \theta$ of -2.7. In other words, a permanent and unanticipated one percentage point decline in the interest rate increases Pareto inequality by 2.7 log points. Alternative calibrations for duration imply values between -2.9 and -2.4. Table 5 in Appendix A contains our estimates for each individual.

Table 3: Measuring the effect of interest rate on tail inequality $\partial_r \log \theta$

	Obs.	Average	Percentiles				
			Min	p25	p50	p75	Max
Baseline	72	−2.7	−18.8	−3.5	−2.0	−0.7	4.1
Duration = 20 years	72	−2.9	−15.4	−3.5	−2.0	−1.4	2.1
Duration = 40 years	72	−2.4	−22.3	−3.4	−0.3	0.2	6.1

Beyond the top 100. Our empirical analysis focuses on the very top of the wealth distribution (i.e., the wealthiest 100 individuals). This is for two reasons. First, it is consistent with

[19]See also Weber (2018) and van Binsbergen (2020).

the theory: our sufficient statistic (Equation 21) states that the effect of interest rates on Pareto inequality depends on its effect on the most extreme wealth trajectories. Second, there is comparatively more available data on individuals at the very top, who tend to own well-known companies.

Despite these data limitations, we argue in Appendix E that our mechanism also applies well beyond the top 100. This is for two reasons. First, roughly half of individuals the top 1% of the wealth distribution actively manage a firm that they founded. Evidence from the Survey of Consumer Finances suggests that these private firms tend to rely heavily external financing. Second, as mentioned in Section 2, our mechanism does not apply only to entrepreneurs who founded firms, but also to any household with a concentrated exposure to a firm that relies on external financing. This includes a large fraction of workers in VC-backed firms. We refer the reader to Appendix E for more details.

4.2 Can the secular decline in discount rates account for the rise in inequality?

Discount rates have been declining steadily since the 1980s, a period that saw a sharp rise in top wealth inequality. We now use our sufficient statistic to quantify the contribution of declining rates on the rise in Pareto inequality in the U.S. over the 1985-2015 period.

Decline in discount rates. In our model, the interest rate r represents the discount rate (i.e., expected return) on an all-equity firm. In the data, the right counterpart for r is the sum of the risk-free rate and the unlevered equity risk premium (ERP).[20] Moreover, the discount rate needs to be adjusted by the growth rate of the economy, which is normalized to be zero in our model.[21]

Real yields on U.S. treasuries have declined steadily since the 1980s. One reference for the change in the real interest rate is the change in the real yield on 10-year U.S. bonds. Figure 3 plots the 10-year annual real yield (i.e., nominal yield net of expected inflation) as estimated by the Cleveland Fed. From 1985 to 2015, the decline is roughly 5 percentage points, from 5.5% to 0.5%.

Estimating the equity risk premium (ERP) is notoriously difficult and there is no consensus in the literature regarding the trend properties of the ERP over the period. For example, Campbell and Thompson (2008) and Martin (2017) find that it has been roughly constant, Duarte and Rosa (2015) and Farhi and Gourio (2018) estimate that it has increased while Greenwald et al.

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[20]To be even more precise, the right discount rate would be the one used to discount high-duration firms, which may be different from the one used for the equity of the representative firm (Gormsen and Lazarus, 2019).

[21]In Appendix E, we introduce per-capita growth γ to the stylized model and show that, in general equilibrium, a change in γ moves the interest rate one-for-one without having any effect on the Pareto exponent.

(a) Decline in risk-free rate
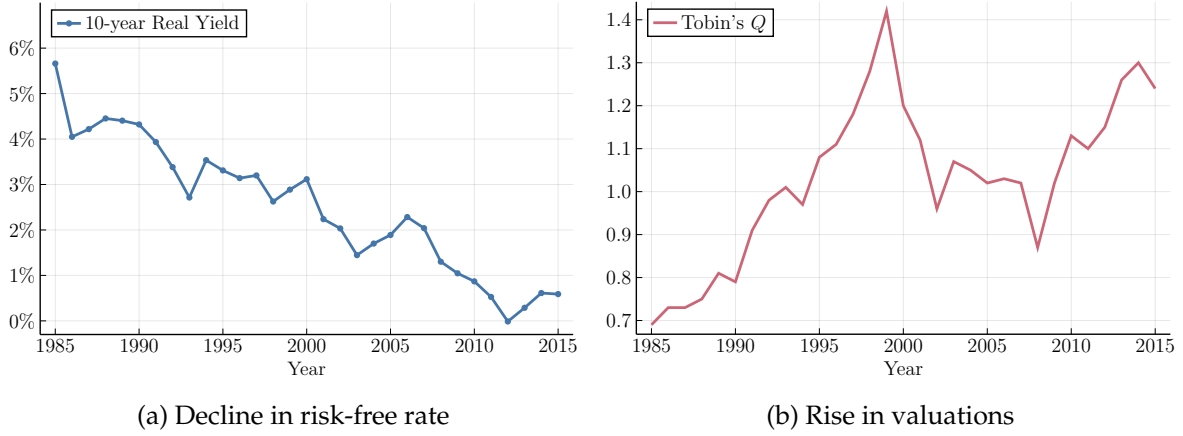
(b) Rise in valuations

Figure 3: Decline in discount rates and rise in valuations (1985-2015)

Notes. "Risk-free rate" is the 10-year U.S. treasury yield minus the 10-year expected inflation as estimated by the Cleveland Fed. We estimate the aggregate market-to-book ratio using data from the Financial Accounts as the ratio of "corporate equities" (BOGZ1LM103164103A) plus total liabilities (BOGZ1FL104190005A) over total assets (NCBTSTA027N) amongst nonfinancial corporate businesses.

(2019) estimate that it has decreased. To be conservative, we consider an ERP that increased from 3.5% to 4.5%. With an average leverage of public firms of 1.5, this roughly corresponds to a change in unlevered ERP from 2.5% to 3%. Finally, we estimate that the growth rate per capita has declined from roughly 2% to 1.5% (see Appendix E for our methodology). Putting these estimates together, we estimate a decline in the growth-adjusted real discount rate r of roughly 4 percentage points, from 6% to 2%.[22]

An alternative way to estimate the decline in discount rates is to look directly at the rise in firms valuations. Figure 3 plots the aggregate market-to-book ratio (i.e., Tobin's Q) in the non-financial corporate sector, which has increased from roughly 0.7 to 1.25 over the 1985-2015 period. This rise is consistent with a decrease in r of 4 percentage points if the duration of the representative firm is $\log(1.25/0.7)/4\% \approx 15$ years.[23]

Rise in Pareto inequality. Figure 4a plots the evolution of top wealth shares in the U.S. using data from Piketty et al. (2018). The top 0.01% wealth share increased more than the top 0.1% share, which itself increased more than the top 1% share. This pattern is a signature of a thickening of the right tail of the wealth distribution (i.e., an increase in Pareto inequality). As discussed in Jones and Kim (2016), Pareto inequality is directly related to the ratio of top shares. Denoting $S(p)$ to be the share of wealth owned by individuals in the top $p \in (0,1)$ quantile of a distribution with Pareto inequality θ, we have:

$$\lim_{p \to 0} \log \frac{S(p)}{S(10p)} = (\theta - 1) \log 10. \tag{26}$$

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[22]See van Binsbergen (2020) for related results.

[23]As mentioned earlier, Gormsen and Lazarus (2019) estimates a duration of roughly 20 years for firms in CRSP.

24

Figure 4b plots estimates of Pareto inequality θ using Equation 26 for $p = 0.1\%$ and $p = 0.01\%$. The two estimates are consistent, and, overall, we find that Pareto inequality increased by roughly $\log(0.7/0.6) = 15$ log points over the 1985-2015 period.
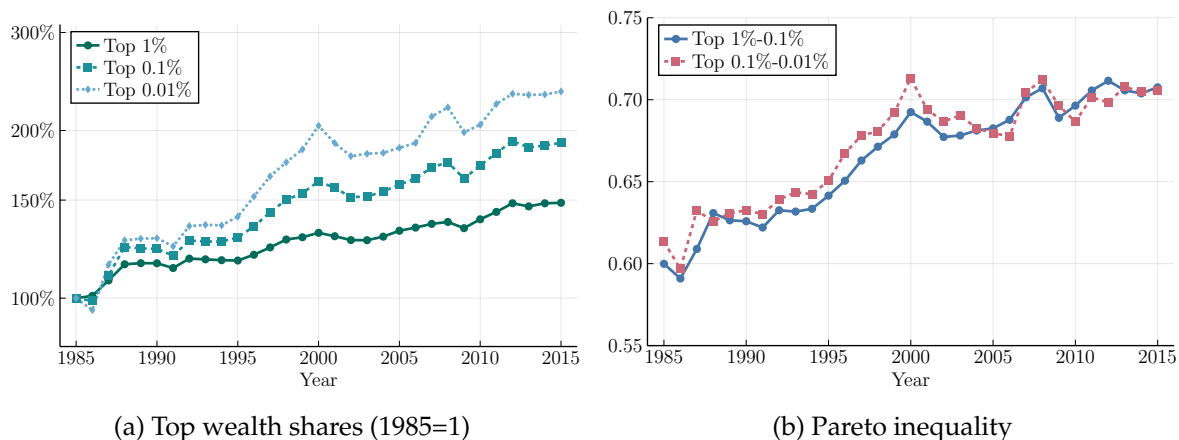

(a) Top wealth shares (1985=1)   (b) Pareto inequality

Figure 4: Rise in top wealth inequality (1985-2015)

Notes. Panel (a) plots top wealth shares divided by their 1985 level. The data is obtained from the replication material of Piketty et al. (2018). The top shares correspond to the share of total household wealth held by a group (equal-split individuals, 20+). Panel (b) plots corresponding estimates for Pareto inequality θ, using Equation 26.

Magnitudes. Given our baseline assumption that the discount rate r has declined by 4 percentage points, our sufficient statistic indicates that the contribution of declining discount rates on Pareto inequality is:

$$\widehat{\partial_r \log \theta} \times (r_{2015} - r_{1985}) \approx -2.7 \times -4\%,$$

$$\approx 11 \text{ log points.} \tag{27}$$

Because the overall change in Pareto inequality during the period was roughly 15 log points (see Figure 4b), our sufficient statistic implies that declining discount rates account for roughly two-thirds of the rise in Pareto inequality.

Endogenous discount rates. Many explanations have been proposed for the recent decline in discount rates, including a shortage of safe assets (Caballero et al., 2008), changing demographics (Carvalho et al., 2016), a shift in monetary policy regime (Lettau et al., 2018), and secular stagnation (Eggertsson et al., 2019).

Throughout the paper, we remain agnostic regarding which factors drive the decline in discount rates. It could come from factors exogenous to our model: for instance, a decline in the subjective discount factor of "workers", as seen in Appendix 2. It could also come from a change in parameters that directly matter for Pareto inequality: for instance, a decline in the

subjective discount factor of entrepreneurs ρ, which governs their saving rate. In this case, our sufficient statistic gives the *partial* derivative of Pareto inequality with respect to the interest rate. More precisely, the total effect of a change of ρ on Pareto inequality θ can be decomposed into a direct and an indirect effect:

$$\frac{\mathrm{d}\log\theta}{\mathrm{d}\rho} = \frac{\partial\log\theta}{\partial\rho} + \underbrace{\frac{\partial\log\theta}{\partial r}}_{\text{sufficient statistic}} \frac{\partial r}{\partial\rho}. \tag{28}$$

The first term corresponds to the direct increase of Pareto inequality due to the higher saving rate of entrepreneurs. The second term corresponds to the indirect increase of Pareto inequality due to lower discount rates in equilibrium. Our sufficient statistic approach allows us to quantify the indirect effect due to a lower equilibrium discount rate, without taking a stance on the reason why discount rates declined.

5 Transition dynamics

Our sufficient statistic approach allows us to transparently quantify the long-run effect of lower rates on Pareto inequality. However, it does not provide any information on the dynamics of the wealth distribution along the transition path. In this section, we simulate the evolution of the wealth distribution to examine these transition dynamics.

This exercise is helpful for at least two reasons. First, Gabaix et al. (2016) argue that the convergence of the right tail of the wealth distribution can be extremely slow, so it is important to verify whether our mechanism is consistent with the fast transition that we observe in the data (see Figure 4a). Second, a surprise decline in future rates leads to high realized returns today: while this effect does not matter in the long-run, it can play a large role in the dynamics of top wealth shares in the short-run. We use an accounting framework to decompose the dynamics of top wealth shares into a within term (due to the growth rate of existing fortunes) and a displacement term (due to the arrival of new fortunes in top percentiles).

Parameters. We use the stylized model from Section 2, augmented with leverage (as described in Section 3). We set the rate of population growth to $\eta = 2\%$ and the subjective discount factor of entrepreneurs to 6%. The model has 4 remaining parameters: the growth rate of the firm (i.e., the tree) g, the investment rate i, the transition rate to rentier (i.e., the rate at which trees blossom) δ, and the debt-to-equity ratio κ.

We choose these remaining 4 parameters to match 5 moments. The first moment is the average Pareto inequality during the 1985-2015 time period. The other four moments are the

ones used to construct the sufficient statistic in section 4: the average growth rate of agents reaching the top of the wealth distribution, as well as the equity payout yield, debt-to-market equity, and duration of the firms they own. This ensures that the model matches the sufficient statistic measured in the previous section. We compare these empirical moments to the ones obtained in the model for an interest rate $r = 4\%$, which corresponds to the average rate during the time period (see Figure 3a).

Given that we have more moments than parameters, we pick parameters to minimize the Euclidean distance between normalized moments in the model and in the data. The result of the calibration is shown in Table 4. Although our model is very stylized, we obtain a satisfactory fit for the targeted empirical moments.

Table 4: Calibration

	Model	Data
Parameters		
Growth rate of firm (g)	39%	
Investment rate (i)	-29%	
Transition rate to rentier (δ)	40%	
Debt-to-equity ratio (κ)	2.8	
Moments		
Pareto inequality	0.73	0.65
Growth rate of wealth	31%	32%
Equity payout yield	-2%	-2%
Debt-to-market equity	0.51	0.48
Duration	29	30

Experiment. We consider a decrease in the interest rate from 6% to 2% over a 30 year period, which mimics the behavior of our estimated growth-adjusted discount rate series, as discussed in the previous section. We assume that the sequence of interest rate changes is unanticipated: individuals take the current interest rate as the permanent one. In other words, we feed in a sequence of "MIT shocks".[24]

Figure 5a plots discount rates as well as realized returns along the transition path. Interest rates decline linearly during the time period. Note, however, that realized returns remain high over the transition period: this is because surprise declines in future rates increase realized returns today.

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[24]An alternative method would be to calibrate the general equilibrium version of the model, as discussed in 2, and shock the model with a linear decrease in the subjective discount factor of workers. As a result, the equilibrium interest rate decreases, although in a non-linear fashion. We have tried implementing this approach and have obtained similar results.

Wealth distribution. To simulate the evolution of the wealth distribution, we time-discretize the model at the weekly frequency and solve for the transition dynamics over a finite grid using the "Pareto extrapolation" developed in Gouin-Bonenfant and Toda (2020). We assume that the wealth distribution is in steady-state at time $t \leq 0$ (i.e., the steady-state associated with the initial interest rate $r = 6\%$). We consider the wealth distribution among entrepreneurs and rentiers.

Figure 5b plots the evolution of the top wealth shares along the transition path. The dynamics of top wealth shares is consistent with the data. In particular, the top 0.01% increases more than the top 0.1%, which in turn increases more than the top 1% (i.e., Pareto inequality increases). We conclude that a model calibrated to match our micro-level evidence generates transition dynamics in line with the data (see Figure 4a). As in Gabaix et al. (2016), the fact that at least on type of agents reaches the right tail of the distribution quickly ensures that our model generates a high convergence speed. In our case, it is successful entrepreneurs who reach the right tail quickly.

(a) Interest rate and realized returns   (b) Top wealth shares
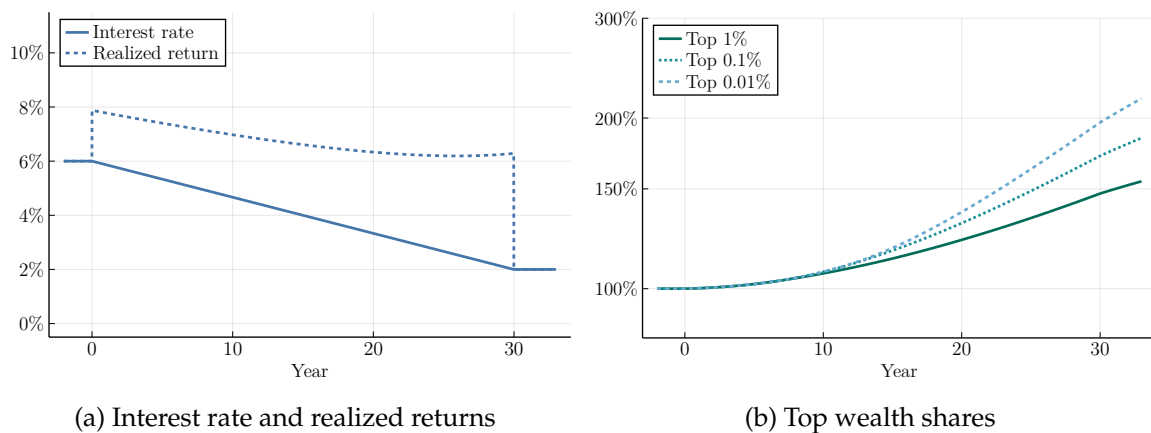
Figure 5: Returns and inequality (model)

Decomposing the rise in top wealth shares. Figure 6 plots the annual growth rate of the top 1% share. The growth rate of the the top 1% wealth share is initially zero since the initial distribution is in a steady state. It then becomes positive during the transition period, consistent with the fact that Pareto inequality increases when the interest rate decreases. In the long run, it converges back to zero, reflecting the fact that the wealth distribution reaches its new steady state.

We now use the accounting framework proposed by Gomez (2019) to examine more closely the transition dynamics of the top 1% wealth share. The framework decomposes the growth of top shares between two time periods into three terms: a within term, a displacement term, and a demography term. The within term captures the growth rate of existing households in the

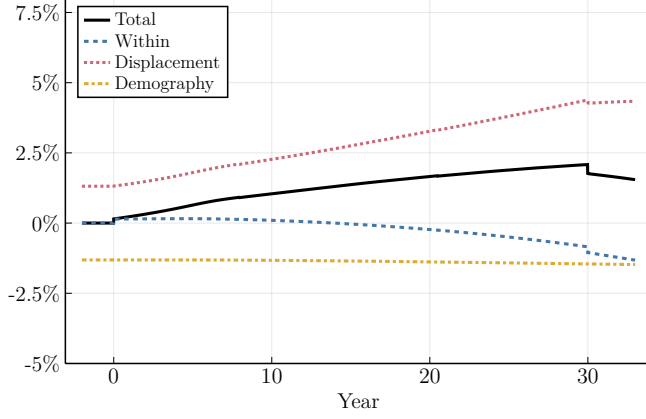Figure 6: Decomposing the rise in the top 1% wealth share (model)

top percentiles relative to the growth rate of aggregate wealth. The displacement term, which is always positive, accounts for the dispersion of realized returns among top individuals. It is the sum of an entry term (the wealth of individuals entering the top percentiles minus the wealth of individuals at the percentile threshold) and an exit term (the wealth of individuals at the percentile threshold minus the wealth of individuals exiting the top percentiles). Finally, the demography term, which is always negative, accounts for population growth. It is the difference between the wealth of individuals entering the top percentiles due to population growth minus the change in aggregate wealth due to population growth.[25]

Figure 6 plots the three components in the accounting decomposition, constructed using our simulated wealth distribution. In steady state, the within term is simply the expected growth rate of individuals relative to the growth rate of the economy (i.e., $r - \rho$). As the interest rate r decreases from 6% to 2%, the within term decreases from 0% to -4%. Along the transition path, however, the within term initially increases due to the fact that entrepreneurs have a higher exposure to realized return than rentiers (because their firm is levered) and that

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[25]Formally, denote T the set of individuals in the top percentiles at time t, T' the set of individuals in the top percentiles at time t', E the set of individuals that enter the top percentiles between t and t', and X the set of individuals that exit the top percentiles between t and t'. The growth of the top share S_t between t and t' can be written as:

$$\frac{S_{t'} - S_t}{S_t} = \underbrace{\frac{\overline{w}_{T,t'} - \overline{w}_{T,t}}{\overline{w}_{T,t}}}_{\text{Within}} + \underbrace{\frac{\frac{|E|}{|T'|}\left(\overline{w}_{E,t'} - \underline{w}_{t'}\right) + \frac{|X|}{|T'|}\left(\underline{w}_{t'} - \overline{w}_{X,t'}\right)}{\overline{w}_{T,t}}}_{\text{Displacement}} + \underbrace{\frac{\frac{|T'|-|T|}{|T'|}\left(\underline{w}_{t'} - \frac{\overline{w}_{T,t'}}{\overline{w}_{T,t}}\overline{w}_{T,t}\right)}{\overline{w}_{T,t}}}_{\text{Demography}}, \tag{29}$$

where $\underline{w}_t$ denotes the wealth of the individual at the percentile threshold (i.e., the 1% quantile) and, for any set Ω, $\overline{w}_\Omega$ denotes the average wealth of individuals in Ω (both normalized by per-capita wealth).

Applying analytical results of Gomez (2019) to our particular setup, we have closed-form formula for the decomposition in a steady-state, as the top percentile tends to zero:

$$0 = \underbrace{r - \rho}_{\text{Within}} + \underbrace{\pi_E \delta \left(\theta \left(\frac{1}{(q + \kappa(q-1))^{1/\theta}} - 1 \right) - \left(\frac{1}{q + \kappa(q-1)} - 1 \right) \right)}_{\text{Displacement}} \quad \underbrace{-\eta\theta}_{\text{Demography}}, \tag{30}$$

where π_E denotes the proportion of entrepreneurs in the right tail of the distribution, given in Proposition 6.

29

entrepreneurs are over-represented in the top 1% of the wealth distribution.

In contrast, the displacement term increases over the transition, from 1.3% to 5.5%. This reflects the fact that, in a low rate environment, successful entrepreneurs grow faster, which increases the flow of new fortunes in the top percentiles. Note that the increase in the displacement term is gradual. This is for two reasons. First, it comes from the fact that the decline in discount rates is itself gradual. Second, and more importantly, even when interest rates are at their lowest level, it takes time for fast-growing entrepreneurs to fully reach the right tail of the distribution.

To summarize the results of our experiment, we find that the rise in the top 1% wealth share is not driven by the within term, but rather by the displacement term. This is consistent with the empirical evidence in Gomez (2019) and Zheng (2019), who show that the recent rise in top wealth shares is not driven by the growth rate of existing fortunes, but rather by the rapid rise of new fortunes. Relatedly, Saez and Zucman (2016) and Hubmer et al. (2020) emphasize the fact that, since 1985, wealth inequality has increased more than what can be expected from realized stock market returns.

6 Conclusion

This paper studies the effect of interest rates on top wealth inequality. We make three distinct contributions. First, we clarify the role of interest rates on top wealth inequality: lower rates *increase* top wealth inequality as long as individuals reaching the top of the wealth distribution are "net borrowers" rather than "net lenders". Second, we derive a sufficient statistic for the effect of lower rates on top wealth inequality (as measured by the Pareto exponent of the wealth distribution). It depends on three key moments: the average growth rate of individuals reaching the top, the average payout yield of the firms that they own, as well as their leverage. Third, we collect new data on the wealth trajectory of the top 100 wealthiest individuals in the U.S., which we use to estimate our sufficient statistics.

Overall, our results indicate that the direct effect of lower rates on top wealth inequality is large: the 4% decline in discount rates from 1985 to 2015 accounts for roughly two-thirds of the rise in top wealth inequality. This finding is guided by the observation that, in the U.S., entrepreneurs reaching the top of the wealth distribution rely heavily on external financing. Technology and institutions presumably affect the extent to which successful firms rely on external financing. In particular, the effect of interest rates on top wealth inequality may be drastically different across countries and time periods. We view our sufficient statistic approach as a first step in understanding this heterogeneous effect.

Finally, our paper complements a growing literature which argues that high wealth inequality puts downward pressure on equilibrium rates of return (see, for instance, Mian et al., 2019 for the interest rate or Gollier, 2001; Toda and Walsh, 2019; Gomez, 2016 for the equity risk premium). In these models, the long-run effect of an "inequality shock" on asset prices is limited: after a rise in inequality, rates decrease, which ultimately dampens the rise in wealth inequality.[26] Our paper argues instead that declining interest rates can further increase top wealth inequality. This suggests that high wealth inequality and low rates can be mutually reinforcing, an idea we leave for future research.

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[26]Formally, the indirect effect, as defined in Equation 28, is negative.

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Online Appendix (not for publication)

A List of top 100 individuals in the 2015 Forbes list

Table 5: List of individual statistics (1-50)

Rank	Name	Type	Public	Equity payout yield	Debt-to-market equity	Growth rate
1	Gates, William		✓	0.00	0.11	0.28
2	Buffett, Warren	financier				
3	Ellison, Lawrence		✓	−0.01	0.20	0.28
4	Bezos, Jeffrey		✓	−0.18	0.22	0.55
5	Koch, David			0.01	0.48	0.12
6	Koch, Charles			0.01	0.48	0.12
7	Zuckerberg, Mark		✓	−0.13	0.03	1.05
8	Bloomberg, Michael			0.01	0.48	0.32
9	Walton, Jim		✓	0.00	0.35	0.22
10	Page, Larry		✓	−0.09	0.05	0.67
11	Brin, Sergey		✓	−0.09	0.05	0.67
12	Walton, Alice		✓	0.00	0.35	0.21
13	Walton, S		✓	0.00	0.35	0.21
14	Walton, Christy		✓	0.00	0.35	0.21
15	Adelson, Sheldon		✓	−0.02	0.57	0.40
16	Soros, George	financier				
17	Knight, Philip		✓	0.01	0.21	0.18
18	Mars, Forrest			0.01	0.48	0.07
19	Mars, John			0.01	0.48	0.07
20	Mars, Jacqueline			0.01	0.48	0.07
21	Ballmer, Steven		✓	0.01	0.11	0.36
22	Icahn, Carl	financier				
23	Dell, Michael		✓	−0.01	0.72	0.33
24	Jobs, Laurene		✓	−0.05	0.33	0.26
25	Chambers, Anne			0.01	0.48	0.05
26	Allen, Paul		✓	0.00	0.11	0.24
27	Blavatnik, Leonard			0.01	0.48	0.36
28	Ergen, Charles		✓	−0.02	0.89	0.50
29	Dalio, Ray	financier				
30	Bren, Donald	financier				
31	Johnson, Abigail	financier				
32	Simons, James	financier				
33	Peterffy, Thomas	financier				
34	Musk, Elon		✓	−0.15	0.27	0.87
35	Soon-Shiong, Patrick		✓	−0.08	0.00	0.58
36	Perelman, Ronald	financier				
37	Cohen, Steven	financier				
38	Schwarzman, Stephen	financier				
39	Tepper, David	financier				
40	Murdoch, Keith		✓	−0.02	1.01	0.27
41	Paulson, John	financier				
42	Beal, Andrew	financier				
43	Anschutz, Philip	rentier				
44	Butt, Charles			0.01	0.48	0.05
45	Newhouse, Donald			0.01	0.48	0.07
46	Newhouse, Samuel			0.01	0.48	0.07
47	Taylor, Jack			0.01	0.48	0.14
48	Schmidt, Eric		✓	−0.02	0.05	0.93
49	Menard, John			0.01	0.48	0.15
50	Kennedy, James			0.01	0.48	0.04

Table 6: List of individual statistics (50-100)

Rank	Name	Type	Public	Equity payout yield	Debt-to-market equity	Growth rate
51	Parry-Okedon, Blair			0.01	0.48	0.04
52	Kinder, Richard		✓	−0.08	1.44	0.56
53	Kaiser, George		✓	−0.01	3.68	0.40
54	Frist, Thomas		✓	−0.01	0.91	0.18
55	Moskovitz, Dustin		✓	−0.13	0.03	0.92
56	Lauder, Leonard		✓	0.02	0.21	0.11
57	Omidyar, Pierre		✓	−0.04	0.18	0.52
58	Meijer, Hank			0.01	0.48	0.08
59	Johnson, Edward	financier				
60	Arison, Micky		✓	−0.01	0.38	0.20
61	Goodnight, James			0.01	0.48	0.23
62	Koum, Jan		✓	−0.07	0.00	1.56
63	Wexner, Leslie		✓	−0.01	0.42	0.16
64	Kroenke, Stanley		✓	0.01	0.35	0.29
65	Broad, Eli	financier				
66	Kohler, Herbert			0.01	0.48	0.03
67	Hamm, Harold		✓	0.00	0.46	0.17
68	Malone, John		✓	−0.06	1.61	0.19
69	Griffin, Kenneth	financier				
70	Geffen, David		✓	−0.03	0.03	0.46
71	Ross, Stephen	financier				
72	Schwab, Charles	financier				
73	Hildebrand, Jeffrey			0.01	0.48	0.37
74	Moore, Gordon		✓	−0.07	0.16	0.18
75	Lauren, Ralph		✓	0.00	0.25	0.17
76	Washington, Dennis	financier				
77	LeFrak, Richard	financier				
78	Rowling, Robert			0.01	0.48	0.37
79	Halle, Bruce			0.01	0.48	0.14
80	Kalanick, Travis		✓	−0.17	0.00	1.53
81	Duffield, David		✓	−0.07	0.11	0.98
82	Cook, Carl			0.01	0.48	0.15
83	Hunt, Ray			0.01	0.48	0.08
84	Khan, Shahid			0.01	0.48	0.25
85	DeVos, Richard			0.01	0.48	0.14
86	Warren, Kelcy		✓	0.03	1.41	0.48
87	Lerner, Theodore	financier				
88	Ilitch, Michael			0.01	0.48	0.14
89	Green, David			0.01	0.48	0.19
90	Keinath, Pauline			0.01	0.48	0.02
91	Johnson, Charles	financier				
92	Rees-Jones, Trevor			0.01	0.48	0.45
93	Kovner, Bruce	financier				
94	Ziff, Robert	rentier				
95	Avara, Dannine		✓	0.02	0.60	0.17
96	Jones, Jerral			0.01	0.48	0.18
97	Williams, Randa		✓	0.02	0.61	0.17
98	Redstone, Sumner		✓	0.01	0.80	0.08
99	Ziff, Dirk	rentier				
100	Ziff, Daniel	rentier				

B Appendix for Section 2

B.1 Proofs

Proof of Proposition 2. Let W denote wealth and $p_R(W), p_E(W)$ denote, respectively, the wealth density of rentiers and entrepreneurs in steady state, normalized so that $\int (p_E(W) + p_R(W))\, dW = 1$. Denoting $\phi(\cdot)$ to be the Dirac function, the Kolmogorov Forward Equations in steady-state are

$$0 = -\partial_W \left(-\frac{i}{q} + g - \rho \right) p_E(W) - (\delta + \eta) p_E(W) + \eta \phi(W - q),$$

$$0 = -\partial_W (r - \rho) p_R(W) - \eta p_R(W) + \delta \frac{1}{q} p_E \left(\frac{W}{q} \right).$$

In this stylized model, we can solve for the stationary solution in closed form. Denote

$$\theta_R = \frac{r - \rho}{\eta}, \quad \theta_E = \frac{-\frac{i}{q} + g - \rho}{\eta + \delta}.$$

If $\theta_E > 0$, we have that

$$p_E(W) = \frac{\eta}{\delta + \eta} \frac{1}{\theta_E} q^{\frac{1}{\theta_E}} W^{-\frac{1}{\theta_E} - 1} 1_{W \geq q},$$

$$p_R(W) = \frac{\delta}{\delta + \eta} \frac{1}{\theta_E - \theta_R} \left(W^{-\frac{1}{\theta_E} - 1} 1_{W \geq 1} - W^{-\frac{1}{\theta_R} - 1} 1_{W \geq 1} 1_{\theta_R > 0} + W^{-\frac{1}{\theta_R} - 1} 1_{W \leq 1} 1_{\theta_R < 0} \right).$$

Otherwise, if $\theta_E < 0$, we have:

$$p_E(W) = -\frac{\eta}{\delta + \eta} \frac{1}{\theta_E} q^{\frac{1}{\theta_E}} W^{-\frac{1}{\theta_E} - 1} 1_{W \leq q}$$

$$p_R(W) = -\frac{\delta}{\delta + \eta} \frac{1}{\theta_E - \theta_R} \left(W^{-\frac{1}{\theta_E} - 1} 1_{W \leq 1} - W^{-\frac{1}{\theta_R} - 1} 1_{W \leq 1} 1_{\theta_R < 0} + W^{-\frac{1}{\theta_R} - 1} 1_{W \geq 1} 1_{\theta_R > 0} \right)$$

Note that θ_R is increasing in r while θ_E is decreasing in r. Moreover, there is a unique $r^* \in (g - \delta, \rho + \delta)$ at which the two functions intersect, which is given by:

$$\theta_E(r^*) = \theta_R(r^*)$$

$$\Leftrightarrow \frac{r^* + \delta \left(1 - \frac{1}{q} \right) - \rho}{\eta + \delta} = \frac{r^* - \rho}{\eta}$$

$$\Leftrightarrow 1 - \frac{1}{q(r^*)} = (r^* - \rho) \frac{\delta}{\eta}$$

$$\Leftrightarrow \eta(-i - r^* - g) = (r^* - \rho)(-i + \delta)$$

$$\Leftrightarrow r^* = \frac{\eta}{-i + \delta + \eta} (g - \delta) + \frac{-i + \delta}{-i + \delta + \eta} (\rho + \eta)$$

The assumption that $\rho < g - i$ ensures that $r^* > \rho$ (i.e., $\theta_R(r^*) = \theta_E(r^*) > 0$). This implies that $\max(\theta_E(r), \theta_R(r))$ is positive on $(g - \delta, \rho + \delta)$.

$\square$

B.2 Fraction of entrepreneurs at the top

Denote $\pi_E(W)$ to be the mass of entrepreneurs relative to the total mass of agents at a wealth level W.

Corollary 6. *We have that*

$$\lim_{W \to +\infty} \pi_E(W) = \begin{cases} \dfrac{1}{1 - \dfrac{\delta}{\eta} q^{-\frac{1}{\theta}} \dfrac{\theta}{\theta - \frac{r-\rho}{\eta}}} & \text{for } r \in (g - \delta, r^*) \\[2em] 0 & \text{for } r \in (r^*, \rho + \eta) \end{cases}$$

Proof of Corollary 6. This follows from the expressions of the wealth density given in the proof of Proposition 2. If $r > r^*$, the relative proportion of entrepreneurs at wealth level W tends to zero as wealth tends to infinity. Otherwise, using the same notations as in the proof of Proposition 2, it tends to

$$\lim_{W \to +\infty} \pi_E(W) = \frac{\eta q^{\frac{1}{\theta_E}} / \theta_E}{\eta q^{\frac{1}{\theta_E}} / \theta_E + \delta / (\theta_E - \theta_R)},$$

$$= \frac{1}{1 + \dfrac{\delta}{\eta} q^{-\frac{1}{\theta_E}} \dfrac{\theta_E}{\theta_E - \theta_R}}.$$

$\square$

B.3 Closing the model

Agents. Suppose that the economy now also includes "workers". Workers have log utility with a subjective discount factor ρ_W. Like entrepreneurs, workers are also born with firms, but they can immediately sell them to the market. Denote π the proportion of agents that are born entrepreneurs.

Market clearing. Denote x to be the steady-state fraction of aggregate wealth owned by entrepreneurs and rentiers (as opposed to workers). Market clearing for goods requires the amount of goods consumed to be equal to the output of maturing trees net of investment:

$$(x\rho + (1 - x)\rho_W) q = \delta - i. \tag{31}$$

Substituting out q in terms of r using Equation 2, we obtain an equation that gives the market clearing interest rate r as a function of x:

$$r = x\rho + (1 - x)\rho_W + g - \delta. \tag{32}$$

State variables. Denote K the total quantity of tree divided by total population. The time derivative of K, denoted $\dot{K}$, is given by:

$$\dot{K} = (g - \delta)K + \eta(1 - K). \tag{33}$$

The time derivative of x, denoted $\dot{x}$, is given by

$$\dot{x} = (r - \rho)x + \eta(\pi \frac{1}{K} - x). \tag{34}$$

The steady-state is characterized by $\dot{x} = \dot{K} = 0$, which gives, combining (33) and (34):

$$(r - \rho - \eta)x = \pi(g - \delta - \eta). \tag{35}$$

Equation 35 and Equation 32 gives a system of two equations and two unknowns x and r. Moreover, there is one and only one solution to the system such that $x \in (0, 1)$.

The next proposition shows that, when π is close enough to zero (i.e., entrepreneurs account for a small share of the total population), changes in ρ_W can generate the full spectrum of interest rates considered in Proposition 2.

Proposition 7. *Denote $r_\pi(\rho_W)$ the interest rate as a function of the subjective discount factor of workers.*

1. *$r_\pi(\cdot)$ is an increasing function of ρ_W. Moreover, as π tends to one, $r_\pi(\cdot)$ spans the interval $(g - \delta, \rho + \eta)$:*

$$\lim_{\pi \to 0} \lim_{\rho_W \to 0} r_\pi(\rho_W) = g - \delta,$$

$$\lim_{\pi \to 0} \lim_{\rho_W \to +\infty} r_\pi(\rho_W) = \rho + \eta.$$

2. *As long as $i < \delta - \frac{\rho}{1 - \frac{g-\delta}{\eta}}$, the distribution of workers always has a thinner tail than the distribution of entrepreneurs or rentiers. In this case, Proposition 2 gives Pareto inequality θ for the full distribution of entrepreneurs, rentiers, and workers.*

Therefore, the proposition says that, when the proportion of workers is high enough in the economy, changes in their subjective discount factor generate changes in the interest rate in the economy

Proof of Proposition 7. There exists one and only one solution $x_\pi(\rho_W) \in (0, 1)$ that solves the system given by Equation 35 and 32:

$$x_\pi(\rho_W) = \begin{cases} \frac{1 + \vartheta(\rho_W) - \sqrt{(1 + \vartheta(\rho_W))^2 - 4\vartheta(\rho_W)\pi}}{2} & \text{if } 0 < \rho_W < \rho \\ \pi & \text{if } \rho_W = \rho \\ \frac{1 + \vartheta(\rho_W) + \sqrt{(1 + \vartheta(\rho_W))^2 - 4\vartheta(\rho_W)\pi}}{2} & \text{if } \rho_W > \rho \end{cases},$$

where $\vartheta(\rho_W) = (\eta - (g - \delta))/(\rho - \rho_W)$.

Moreover, we have that

$$\lim_{\rho_W \to 0} r_\pi(\rho_W) = g - \delta + \rho \frac{1 + \vartheta(0) - \sqrt{(1 + \vartheta(0))^2 - 4\vartheta(0)\pi}}{2},$$

$$\lim_{\rho_W \to +\infty} r_\pi(\rho_W) = \rho + \eta - \pi(\eta - (g - \delta)).$$

Therefore,

$$\lim_{\pi \to 0} \lim_{\rho_W \to 0} r_\pi(\rho_W) = g - \delta,$$

$$\lim_{\pi \to 0} \lim_{\rho_W \to +\infty} r_\pi(\rho_W) = \rho + \eta.$$

Finally, denote $\theta_W(r) = \frac{r - \rho_W}{\eta}$. To ensure that the tail of the wealth distribution is not dominated by workers, a sufficient condition is that

$$\theta_E(r_\pi(\rho)) > \theta_W(r_\pi(\rho)).$$

Since $r_\pi(\rho) = \rho + g - \delta$, the condition can be rewritten as

$$\frac{g - \delta + \delta \left(1 - \frac{1}{q}\right)}{\delta + \eta} > \frac{g - \delta}{\eta} \implies i < \delta - \frac{\rho}{1 - \frac{g - \delta}{\eta}},$$

which concludes the proof. $\qquad\square$

C   Appendix for Section 3

C.1   Proofs

Proof of Proposition 3. Denote $\boldsymbol{p}_{Et}(w)$ the joint density of log wealth w and productivity state for entrepreneurs (an $S \times 1$ vector). Denote $p_{Rt}(w)$ the density of log wealth for rentiers. Moreover, denote $\boldsymbol{m}_{Et}(\xi), m_{Rt}(\xi)$ the corresponding moment generating function for wealth:

$$\boldsymbol{m}_{Et}(\xi) \equiv \int_{-\infty}^{+\infty} e^{\xi w} \boldsymbol{p}_{Et}(w), \quad m_{Rt}(\xi) \equiv \int_{-\infty}^{+\infty} e^{\xi w} p_{Rt}(w).$$

It solves the following system of ODEs:[27]

$$\partial_t \boldsymbol{m}_{Et}(\xi) = \mathcal{D}(\boldsymbol{q})^\xi \left(\xi \mathcal{D}(\boldsymbol{\mu}) + \mathcal{T} - (\delta + \eta)\mathcal{I}\right) \mathcal{D}(\boldsymbol{q})^{-\xi} \boldsymbol{m}_{Et}(\xi) + \eta \mathcal{D}(\boldsymbol{q})^\xi \boldsymbol{\psi}, \qquad (36)$$

$$\partial_t m_{Rt}(\xi) = (\xi(r - \rho) - \eta) m_{Rt}(\xi) + \delta \mathbf{1}' \boldsymbol{m}_{Et}(\xi), \qquad (37)$$

where $\boldsymbol{q} = (q_1, \ldots, q_S)'$ is the vector of prices (i.e., the solution to the HJB, see Equation 8), $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_S)'$ is the distribution of firm types at birth, $\mathcal{D}(\boldsymbol{v})$ is the diagonal matrix with diagonal elements given by the vector $\boldsymbol{v}$, and $\mathcal{I}$ is the identity matrix. Denote $p_t(w)$ the overall density of log wealth. We have $p_t(w) = \mathbf{1}' \boldsymbol{p}_{Et}(w) + p_{Rt}(w)$, which implies that

$$m_t(\xi) = \mathbf{1}' \boldsymbol{m}_{Et}(\xi) + m_{Rt}(\xi).$$

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[27]One way to obtain it is to start from the KFE for the density of book log wealth $b_t \equiv w_t - \log(q_{s_t})$, which is locally deterministic. Multiplying by $\mathcal{D}(q^\xi)e^{\xi b}$ and integrating over b gives Equation 36.

We now focus on characterizing the limit $\lim_{t \to \infty} m_t(\xi)$. First, recall that we assumed that there exists at least one state s such that the rate of capital accumulation is positive (i.e., $\mu_s > 0$), which ensures that there exists a unique $\theta_E > 0$ such that $\rho_D \left(\frac{1}{\theta_E} \mathcal{D}(\boldsymbol{\mu}) + \mathcal{T}' - \mathcal{I}(\delta + \eta) \right) = 0$ (see Remark 3.5 of Beare and Toda, 2017). This allows us to characterize the limit $m_t(\xi)$ as time tends to infinity:

$$\lim_{t \to +\infty} m_t(\xi) = \left(1 + \frac{\delta}{\eta - \xi(r - \rho)} \right) \mathbf{1}' \mathcal{D}(\boldsymbol{q})^\xi \left((\delta + \eta)\mathcal{I} - (\xi \mathcal{D}(\boldsymbol{\mu}) + \mathcal{T}') \right)^{-1} \eta \psi \tag{38}$$

if $0 \leq \xi < \min(\frac{\eta}{r-\rho}, \frac{1}{\theta_E})$, and infinity if $\xi \geq \min(\frac{\eta}{r-\rho}, \frac{1}{\theta_E})$. That is, $\lim_{t \to +\infty} m_t$ has a pole in $\min \left(\frac{\eta}{r-\rho}, \frac{1}{\theta_E} \right)$. Using Theorem 3.1 in Beare et al. (2020), this implies that the long-run wealth distribution has a right Pareto tail with Pareto inequality given by $\theta = \max \left(\frac{r-\rho}{\eta}, \theta_E \right)$. $\square$

Proof of Proposition 4. Denote $\boldsymbol{u}(\theta, r), \boldsymbol{v}(\theta, r)$ the left and right eigenvector associated with the dominant eigenvalue of the matrix $\frac{1}{\theta} \mathcal{D}(\boldsymbol{\mu}) + \mathcal{T}$, normalized so that $\boldsymbol{u}'\mathbf{1} = \boldsymbol{u}'\boldsymbol{v} = 1$. Pareto inequality θ is implicitly characterized by the following equation

$$\left(\frac{1}{\theta} \mathcal{D}(\boldsymbol{\mu}) + \mathcal{T} \right) \boldsymbol{v}(\theta, r) = (\delta + \eta)\boldsymbol{v}(\theta, r). \tag{39}$$

Differentiating this equation with respect to r, we obtain

$$\left(\frac{1}{\theta} \mathcal{D}(\partial_r \boldsymbol{\mu}) - \frac{1}{\theta^2} \mathcal{D}(\boldsymbol{\mu})\partial_r \theta \right) \boldsymbol{v} + \left(\frac{1}{\theta} \mathcal{D}(\boldsymbol{\mu}) + \mathcal{T} \right) (\partial_r \boldsymbol{v} + \partial_\theta \boldsymbol{v} \partial_r \theta) = (\delta + \eta) (\partial_r \boldsymbol{v} + \partial_\theta \boldsymbol{v} \partial_r \theta) . \tag{40}$$

Left-multiplying by the left eigenvector $\boldsymbol{u}$ and re-arranging, we obtain

$$\boldsymbol{u}' \left(\frac{1}{\theta} \mathcal{D}(\partial_r \boldsymbol{\mu}) - \frac{1}{\theta^2} \mathcal{D}(\boldsymbol{\mu})\partial_r \theta \right) \boldsymbol{v} = \underbrace{\left(\boldsymbol{u}' \left(\frac{1}{\theta} \mathcal{D}(\boldsymbol{\mu}) + \mathcal{T} \right) - \boldsymbol{u}'(\delta + \eta) \right)}_{=0} (\partial_r \boldsymbol{v} + \partial_\theta \boldsymbol{v} \partial_r \theta) , \tag{41}$$

since u is the left-eigenvector.

Finally, one can show that $\boldsymbol{u}'\mathcal{D}(\boldsymbol{\mu})\boldsymbol{v} > 0$ since it corresponds to the derivative of $\xi \to \rho_D(\xi \mathcal{D}(\boldsymbol{\mu}) + \mathcal{T})$ at $\xi = 1/\theta$, which is convex (see Beare et al. (2020)). Therefore, we obtain the following expression:

$$\partial_r \log \theta = \frac{\boldsymbol{u}'\mathcal{D}(\partial_r \boldsymbol{\mu})\boldsymbol{v}}{\boldsymbol{u}'\mathcal{D}(\boldsymbol{\mu})\boldsymbol{v}}. \tag{42}$$

This concludes the proof. $\square$

Proof for Proposition 5. Denote, for an agent born at time t_0, the process $f_t = \int_{t_0}^t \partial_r \mu_s \, ds$. Denote $\boldsymbol{p}_t(w, f)$ the cross-sectional density of productivity state s, log wealth w, and f for entrepreneurs. Denote $\boldsymbol{m}_t(w, \lambda) = \int_\mathbb{R} e^{\lambda f} \boldsymbol{p}_t(w, f) \, df$ the moment generating function of f. Finally, denote $\tilde{\boldsymbol{m}}_t(\xi, \lambda) = \int_\mathbb{R} e^{\xi w} \boldsymbol{m}_t(w, \lambda) \, dw$ the joint moment generating function of f and w.

The law of motion for $\tilde{\boldsymbol{m}}_t$ is given by:[28]

$$\partial_t \tilde{\boldsymbol{m}}_t = \mathcal{D}(q)^{\xi} \left(\lambda \mathcal{D}(\partial_r \boldsymbol{\mu}) + \xi \mathcal{D}(\boldsymbol{\mu}) + \mathcal{T}' - (\eta + \delta)\mathcal{I} \right) \mathcal{D}(q)^{-\xi} \tilde{\boldsymbol{m}} + \eta \mathcal{D}(q)^{\xi} \boldsymbol{\psi}. \tag{43}$$

We know that $\xi \to \rho_D \left(\xi \mathcal{D}(\boldsymbol{\mu}) + \mathcal{T} \right) = \delta + \eta$ has a unique positive root given by $\xi = 1/\theta$. This implies that, for λ close enough to zero, $\xi \to \rho_D \left(\lambda \mathcal{D}(\partial_r \boldsymbol{\mu}) + \xi \mathcal{D}(\boldsymbol{\mu}) + \mathcal{T} \right) = \delta + \eta$ has a unique positive solution, that Equation 43 implies that $\xi^*(\lambda)$. $\xi^*(\lambda)$ is a pole for the stationary value of $\tilde{\boldsymbol{m}}$:

$$\tilde{\boldsymbol{m}}(\xi) \equiv \lim_{t \to +\infty} \tilde{\boldsymbol{m}}_t(\xi) \tag{44}$$

$$= \mathcal{D}(q)^{\xi} \left((\eta + \delta)\mathcal{I} - \lambda \mathcal{D}(\partial_r \boldsymbol{\mu}) - \xi \mathcal{D}(\boldsymbol{\mu}) - \mathcal{T}' \right)^{-1} \eta \boldsymbol{\psi} \tag{45}$$

for $0 < \xi < \xi^*(\lambda)$, and infinity if $\xi \geq \xi^*(\lambda)$.

As in Beare et al. (2020), the fact that $\tilde{\boldsymbol{m}}$ has a pole in $\xi^*(\lambda)$ implies that $\boldsymbol{m}$ has a right tail with exponent $\xi^*(\lambda)$. More precisely, for $1 \leq s \leq S$, we have:

$$\log \boldsymbol{m}_s(\lambda, w) \sim \xi^*(\lambda)w \text{ as } w \to +\infty \tag{46}$$

To conclude, we express the expectation of f conditional on being in the right tail in terms of $\log \boldsymbol{m}$:

$$\mathrm{E}\left[f \,|\, \log W = w, s = s \right] = \partial_{\lambda=0} \mathrm{E}\left[e^{\lambda f} \,|\, \log W = w, s = s \right]$$

$$= \frac{\partial_{\lambda=0} \int_{\mathbb{R}} e^{\lambda f} \boldsymbol{p}_s(w, f) \, \mathrm{d}f}{\int_{\mathbb{R}} \boldsymbol{p}_s(w, f) \, \mathrm{d}f}$$

$$= \partial_{\lambda=0} \log \boldsymbol{m}_s(\lambda, w)$$

$$\sim \xi^{*\prime}(0)w \text{ as } w \to +\infty$$

Finally, we can follow the same steps as in the proof of 4 to show that ξ^* is differentiable at 0, with:

$$\xi^{*\prime}(0) = \frac{(\boldsymbol{u} \cdot \boldsymbol{v})' \partial_r \boldsymbol{\mu}}{(\boldsymbol{u} \cdot \boldsymbol{v})' \boldsymbol{\mu}}. \tag{47}$$

This concludes the proof. □

D   Extensions

We now cover two extensions of the general model developed in section 3. We first explore constraints on external financing. We then discuss the case of different utility functions for entrepreneurs.

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[28]One way to obtain it is to start from the KFE for the density of book log wealth $b_t \equiv w_t - \log(q_{s_t})$, f and state. Multiplying by $\mathcal{D}(q^{\xi})e^{\xi b + \lambda f}$ and integrating over b, f gives Equation 43.

D.1 Constraint on external financing

We now consider an extension of the baseline model that includes a constraint on external financing. In particular, we now assume that firms face a constraint of the form

$$(1 + \kappa)(a_s - i(g_s)) - \kappa(r - g_s) \geq \psi_s q_{\kappa s}, \tag{48}$$

where ψ_s is a state-dependent lower bound on the equity payout yield.

The benchmark model can be thought as a special case where $\psi_s = -\infty$ (i.e. no lower bound on the equity payout yield). Another lower bound that is often considered in the literature on firm dynamics and entrepreneurship is $\psi_s = 0$, i.e., the firm can not issue equity. Given our assumption of constant debt to equity $\kappa \geq 0$, the constraint also acts as a limits the amount of debt financing. More generally, an intermediate $-\infty < \psi_s < 0$ acts as a constraint on the amount of equity that can be raised in a given time period.

Incorporating the equity financing constraint, the firm problem (8) becomes

$$rq_{\kappa s} = \max_g \left\{ (1 + \kappa)(a_s - i(g_s)) - \kappa(r - g_s) + q_{\kappa s} g + \left(\mathcal{T} q_\kappa \right)_s \right\} \tag{49}$$

$$\text{s.t.} \quad (1 + \kappa)(a_s - i(g_s)) - \kappa(r - g_s) \geq \psi_s q_{\kappa s}. \tag{50}$$

Denote $\lambda_s \geq 0$ the Lagrange multiplier on the external financing constraint. The first-order condition for optimal investment becomes

$$\underbrace{\left((1 + \kappa)i'(g_s) - \kappa \right)(1 + \lambda_s)}_{\text{marginal cost}} = \underbrace{q_{\kappa s}.}_{\text{marginal benefit}} \tag{51}$$

Given that $\lambda_s \geq 0$, the marginal benefit of investment weakly exceeds its marginal cost. The derivative of the wealth drift μ_s is now given by:

$$\partial_r \mu_s = \frac{(1 + \kappa)(a_s - i(g_s)) - \kappa(r - g_s)}{q_{\kappa s}} |\partial_r \log q_{\kappa s}| - \frac{\kappa}{q_{\kappa s}} + \left(1 - \frac{(1 + \kappa)i'(g_s) - \kappa}{q_{\kappa s}} \right), \tag{52}$$

$$\partial_r g_s = \frac{(1 + \kappa)(a_s - i(g_s)) - \kappa(r - g_s)}{q_{\kappa s}} |\partial_r \log q_{\kappa s}| - \frac{\kappa}{q_{\kappa s}} + \left(1 - \frac{1}{1 + \lambda_s} \right) \partial_r g_s. \tag{53}$$

In states s where the constraint does not bind ($\lambda_s = 0$), the last term is zero and we obtain the usual formula. In states s where the constraint binds ($\lambda_s > 0$), the interest rate does not change the equity payout yield ψ_s, and, therefore, we have that $\partial_r \mu_s = \partial_r g_s$. In all cases, we can write:

$$\partial_r \mu_s = (1 + \lambda_s) \left(\frac{(1 + \kappa)(a_s - i(g_s)) - \kappa(r - g_s)}{q_{\kappa s}} |\partial_r \log q_{\kappa s}| - \frac{\kappa}{q_{\kappa s}} \right). \tag{54}$$

In other words, we find that the effect of r on μ_s is the same as in the baseline model (Equation 16), except that it is multiplied by $(1 + \lambda_s)$. Since $1 < i'(g_s) \leq q_s$, the multiplier is bounded: $1 < (1 + \lambda_s) \leq q_{\kappa s}$. The fact that the multiplier is higher than one is intuitive: lower rates relax constraints on external financing, which allows firms to grow faster, thereby increasing the entrepreneurs' rate of capital accumulation.

The upper bound on the multiplier is attained when investment frictions become vanishingly small, i.e. $i'(g_s) \to 1$. In this case, the financing friction is the only force that keeps the growth rate of the firm from being infinite. This case corresponds to the type of firm dynamics modeled in Cagetti and De Nardi (2006) or Moll (2014). In the particular case in which equity issuance is not allowed (i.e., $\psi_s = 0$), Equation 54 gives $\partial_r \mu_s = -\kappa$: the effect on interest rate on Pareto inequality is simply given by the ratio between debt-to-equity and the average growth rate of households reaching the top.

Plugging Equation 54 into Equation 20, we obtain the following expression for the derivative of Pareto inequality with respect to the interest rate:

$$\partial_r \log \theta = \lim_{w \to +\infty} \mathrm{E} \left[(1 + \text{ Lagrange multiplier}) \frac{\text{equity payout yield} \times \text{ duration} - \text{ debt-to-market equity}}{\text{growth rate}} \Big| W = w \right].$$
(55)

D.2 EIS different from one

We now consider an extension of the baseline model that allows for an arbitrary Elasticity of Intertemporal Substitution (EIS) for entrepreneurs. So far we have assumed that entrepreneurs have log utility, which implied that the consumption rate of entrepreneur did not react to the interest rate. In this extension, we consider the case in which entrepreneurs have Epstein-Zin utilities with arbitrary relative risk aversion RRA and elasticity of intertemporal substitution EIS.

The effect of interest rate on the consumption rate is given by the difference between 1 and the EIS. If EIS > 1, the substitution effect is more important than the income effect: households react to a decrease in the interest rate by increasing their consumption rate. If EIS < 1, the income effect is more important than the substitution effect: households react to an decrease in the interest rate by decreasing their consumption rate.

The expression for the effect of rates on the drift of wealth must be adjusted for its effect on the consumption rate of entrepreneurs:

$$\partial_r \mu_s = \frac{(1 + \kappa)(a_s - i(g_s)) - \kappa(r - g_s)}{q_{\kappa s}} |\partial_r \log q_{\kappa s}| - \frac{\kappa}{q_{\kappa s}} + \text{EIS} - 1.$$
(56)

We find that the effect of r on μ_s is the same as in the baseline model (Equation 16), except that there is a new term EIS $- 1$.

Plugging Equation 56 into Equation 20, we obtain the following expression for the derivative of Pareto inequality with respect to the interest rate:

$$\partial_r \log \theta = \lim_{w \to +\infty} \mathrm{E} \left[\frac{\text{equity payout yield} \times \text{ duration} - \text{ debt-to-market equity} + \text{EIS} - 1}{\text{growth rate}} \Big| W = w \right].$$
(57)

A EIS higher than one reduces the effect of lower rates on inequality: this is because, faced with lower future investment opportunities, entrepreneurs start consuming more as a proportion of their wealth. In contrast, a EIS lower than one amplify the effect of lower rates on inequality: due to an income effect, entrepreneurs start consuming less as a proportion of their wealth.

Vissing-Jørgensen (2002) suggests that the EIS of stockholders tends to be close to one, so we maintain this assumption in our empirical results. Our approach, however, would make it easy to explore different values for the EIS.

E Appendix for Section 4

E.1 Beyond the top 100

We separately examine two types of entrepreneurs in the top 1%: workers in VC-backed firms, whose wealth is tied down to their firm, and founders of private firms, which rely on bank lending.

VC-backed firms. Firms backed by VCs are an important part of the US economy. According to Capshare, 10,400 companies received venture funding in 2018.[29] On average, the ownership share of the founders decreases by roughly 25% every funding round. Since funding rounds tend to happen every 18 month, this corresponds to an annual dilution rate of 16% (equity payout yield of -16%).

While the number of VC firms is small relative to the number of households, it is worth noting that many key employees of these firms receive a substantial proportion of their income as equity. Equity compensation typically leads to concentrated portfolios due to a mix of vesting time and other restrictions on stock sales (especially pre-IPO). Our notion of "entrepreneurs" in the model can be interpreted as including not only the founder of the firm, but also any individual who invests the majority of their wealth in the firm. In particular, it also includes employees that receive a substantial proportion of their income as equity. Eisfeldt et al. (2019) reports that equity compensation represents almost 45% of total compensation to high-skilled labor in recent years and that employees working in VC-backed firms account for approximately 2% of the workforce.

Despite the lack of data on the portfolio of such "human capitalists", we think that many wealthy, high-skilled employees have portfolios with concentrated holdings. Almost by definition, we expect this concentrated holdings to be particularly important for firms that are net equity issuers.

Private businesses. To extend our analysis beyond VC-backed firms, we use data from the 2016 wave of the *Survey of Consumer Finances* (SCF) to focus on individuals who founded (or acquired) a formal business which they actively manage.

Table 7 presents summary statistics related to the importance of entrepreneurs in the right tail of the wealth distribution. First, notice that entrepreneurs are over-represented at the top. As in Cagetti and De Nardi (2006), we find that wealthier individuals are much more likely to manage a firm. In the full population, 11% of individuals founded their own firms; in the top 0.01% the fraction increases to 66%. Second, the businesses founded by wealthy individuals tend to be pass-through entities, which is consistent with the evidence in Cooper et al. (2016). For instance, 93% of businesses owned by households in the top 0.01% are partnerships or S corporations. This is in a sharp contrast with the fact that roughly

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[29]Capshare is a "web application that helps businesses manage their stock and assets on one organized platform". All our statistics are taken from their "2018 Private Company Equity Statistics Report".

Table 7: Entrepreneurs in the top percentiles (SCF, 2016)

Groups (shares)	Total	Top 1%	Top 0.1%	Top 0.01%
Entrepreneurs	0.11	0.43	0.58	0.66
Sole proprietorship	0.48	0.09	0.06	0.02
Partnership	0.35	0.60	0.64	0.63
S corporation	0.11	0.21	0.24	0.30
Other corporations	0.06	0.11	0.06	0.05
Others	0.89	0.57	0.42	0.34

two-thirds of entrepreneurs in the top 100 own public firms (i.e., C corporations). This suggests that most of the outside financing for entrepreneurs the top 1% takes the form of debt financing, rather than equity financing.

The effect of interest rates on wealth inequality depends on the extent to which these businesses have used external financing on their way to the top (through either equity financing or debt financing). Due to data limitations, we are unable to produce estimates of the equity issuance and leverage of the firms owned by entrepreneurs in the top 1%. Still, we can use a question in the SCF regarding the use of external financing in the previous year for (i.e., "What external sources of money were used to finance the ongoing operations or improvements in this business during the past year?"). The question is only available for firms with less than 500 employees employees, which account for 80% of private businesses in the top 0.01%.

Table 8: Use of external financing during the last year (SCF, 2016)

	Total	Top 1%	Top 0.1%	Top 0.01%
No external financing	0.84	0.72	0.71	0.77
Debt financing	0.16	0.27	0.28	0.18
Equity financing	0.004	0.004	0.006	0.046

Notes. "Debt financing" includes business loans and personal loans; "Equity financing" only includes equity investment by other investors.

Table 8 contains summary statistics on the use external financing. The key takeaway is that entrepreneurs use both equity and debt financing to grow their firm. In a given year, 27% of entrepreneurs in the top 1% use debt financing while 0.4% use equity financing. One way to interpret these numbers is that, once they are in the top, entrepreneurs in the top 0.01% raise debt once every 5 years and raise equity once every 20 years. These figures may underestimate the extent to which these businesses have relied on outside financing to grow. Indeed, our evidence from top entrepreneurs in Forbes suggests that most of the issuance happens as households are still reaching the top, rather than when they are already at the top.

E.2 Per-capita growth

Per capita output growth. In the stylized model (see section 2), there is no per-capita output growth. Augmenting the model for per capita output growth is straightforward: the interest rate simply in-

creases one to one, and Pareto inequality θ and q are invariant to γ.

We now show this formally. Assume that trees grow at rate $\gamma + g$ and that the initial size of trees grows at rate γ. Therefore, the per-capita output growth is now γ. Market clearing for goods implies that consumption equals output net of investment

$$\rho q = \delta - i.$$

Therefore, q is invariant w.r.t. γ. Since, along a balanced growth path,

$$q = \frac{\delta - i}{r - \gamma - g + \delta},$$

we get that

$$r = \rho + (g - \delta) + \gamma.$$

Therefore, r increases one-for-one with γ.

To obtain an expression for Pareto inequality in the model, we now need to look at the *relative* growth rate of wealth for entrepreneurs and workers (the growth rate of wealth minus aggregate growth rate of wealth γ). Subtracting γ from the expressions in Equations (3) and (4), the expressions become

$$\frac{dW_t}{W_t} = \begin{cases} \left(-\frac{i}{q} + g - \rho \right) dt & \text{if } t < T \\ (r - \rho - \gamma) \, dt & \text{if } t > T. \end{cases}$$

Both relative growth rates are invariant w.r.t. γ. As a result, Pareto inequality is also invariant w.r.t. γ.

Estimating trend per capita output growth. The previous analysis highlights the fact that what determines the level of inequality in the model is not the level of the interest rate per se, but the difference between the interest rate and the growth rate of output per capita. We now describe our procedure to estimate 10-year ahead GDP growth. Using annual data on real GDP y_t from 1947 to 2019. We estimate the following regressions

$$\log y_{t+10} = \mu + \sum_{k=0}^{p} \rho_k \log y_{t-k} + \varepsilon_{t+10},$$

$$\log y_t = \tilde{\mu} + \sum_{k=0}^{p} \tilde{\rho}_k \log y_{t-2-k} + \tilde{\varepsilon}_t.$$

The first equation is a linear forecasting equation of 10-year ahead log GDP on p lags. The second is a nowcasting equation of log GDP on lags from $t - 2$ to $t - p$. We set $p = 4$. The predicted value of the first equation, $E_t \log y_{t+10}$, corresponds to the 10-year ahead forecast of log GDP on the basis of data available at time t. As argued by Hamilton (2018), the predicted value of the second equation, $E_{t-2} \log y_t$, corresponds to the "trend component" of log GDP. Hamilton (2018) argues that this

procedure to extract the trend component compares favorably with the commonly used HP filtering approach.

We define the 10-year ahead trend GDP growth at time t as $g_t \equiv \frac{1}{10} \frac{\mathrm{E}_t y_{t+10} - \mathrm{E}_{t-2} y_t}{\mathrm{E}_{t-2} y_t}$. Using a second order Taylor approximation of the exponential function, we obtain the following estimator:

$$\hat{g}_t = \frac{1}{10} \frac{e^{\widehat{\log y_{t+10}}}\left(1 + \frac{\hat{\sigma}^2}{2}\right) - e^{\widehat{\log y_{t+10}}}\left(1 + \frac{\hat{\hat{\sigma}}^2}{2}\right)}{e^{\widehat{\log y_{t+10}}}\left(1 + \frac{\hat{\hat{\sigma}}^2}{2}\right)},$$

where $\widehat{\log y_{t+10}}$ and $\widehat{\log y_{t+10}}$ are the predicted values from the first and second regressions, respectively. Similarly, $\hat{\sigma}^2$ and $\hat{\hat{\sigma}}^2$ are the mean squared error from the first and second regressions, respectively. In both, the "Jensen gap" $\frac{\sigma^2}{2}$ is negligible (< 0.002).

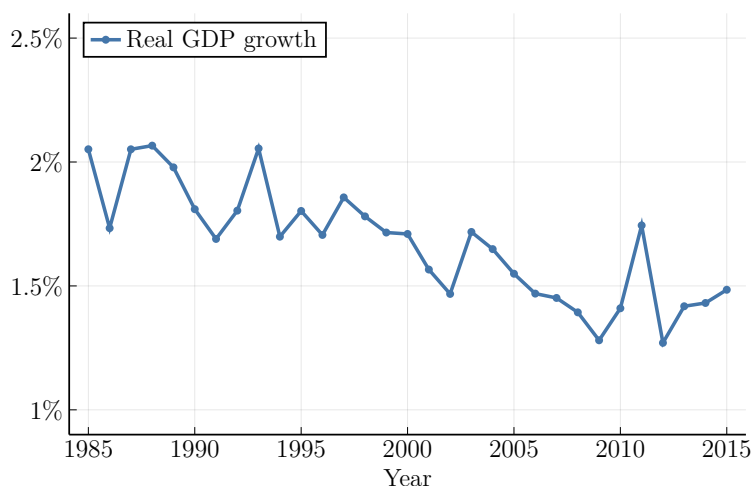We plot our results in Figure 7: we find that the per capita growth rate has declined from 2% to 1.5% over the time period.

Figure 7: Trend real GDP growth per capita